9. Simple and semisimple rings (Après Bourbaki Algebra Chapter 8 §5)

[s:sssrings]

9.1. **Semisimple rings.** [ss:ssrings]

A ring is *semisimple* if it is semisimple as a (left) module over itself, or, equivalently, if every (left) module over it is semisimple. (Every module is a sum of its cyclic submodules. And every cyclic module is a quotient of ${}_AA$.)

Some facts about semisimple rings:

- A semisimple ring has finite length: in other words, it is Artinian and Noetherian. (Any finitely generated semisimple module has finite length.)
- Simple modules of a semisimple ring are precisely its minimal left ideals. (Simple modules are quotients of the ring, but the ring being semisimple every quotient is isomorphic to a sub.)
- For any left ideal $\mathfrak{l}$ of a semisimple ring A, there is an idempotent e such that $\mathfrak{l} = \mathfrak{l}e = Ae$. (Write $A = \mathfrak{l} \oplus \mathfrak{l}'$. The projection to $\mathfrak{l}$ is given by right multiplication by an element e. We have $e^2 = 1e^2 = 1e = e$ and $\mathfrak{l} = Ae = \mathfrak{l}e$.)
- The isotypic components of a semisimple ring are precisely its minimal two-sided ideals. Any two sided ideal is a sum of these. (Characteristic left ideals are precisely two-sided ideals. Thus two-sided ideals are sums of isotypic components. Conversely, isotypic components being characteristic are two-sided ideals.)
- Any quotient of a semisimple ring is semisimple.

[p:oppssring]

Proposition 9.1. *The opposite of a semisimple ring is semisimple.*

Proof. The opposite of a semisimple module being semisimple (Theorem **??**), we know that M^{opp} is semisimple where M denotes a semisimple ring as a left module over itself. But, as is readily verified, M^{opp} is just the ring A^{opp} as a left module over itself (A^{opp} denotes the opposite ring of A). □

[p:ssrings2]

Proposition 9.2. *A ring is semisimple if and only if it is the ring of endomorphisms of a finitely generated semisimple module (over some ring).*

Proof. If A is semisimple, then so is A^{opp}. And the endomorphism ring of A^{opp} is A itself. Conversely, suppose M is a semisimple finitely generated module over a ring A, and let C denote its commutant. From the finite generatedness of M we conclude that C imbeds in a finite direct sum of copies of M^{opp}. But M^{opp} is semisimple. Therefore so is C. □

[ss:srings]

9.2. **Simple rings.**

A semisimple ring satisfying any of the equivalent conditions below is called a *simple ring*. The equivalence of the conditions is easy to see given the facts listed in the previous section about semisimple rings.

- there is only one class of simple submodules;
- the ring is isotypic as a module over itself;
- the only non-trivial two-sided ideal is the whole ring.

We can talk about the *length* of a simple ring: it is the multiplicity of the simple module in the ring (as a left module over itself). This length is finite.

The only commutative simple rings are fields. A division ring is simple. The structure theorem for simple rings in §9.2 says that they are precisely $r \times r$ matrix rings over division rings. We will show later (in §10) the following:

> an Artinian ring whose only non-trivial two-sided ideal is the whole ring is simple.

[ss:scompss]

9.3. **The simple components of a semisimple ring.** Let A be a semisimple ring.

[t:scompss]

Theorem 9.3. *The ring A has only a finite number of minimal two-sided ideals, say A_i, $1 \leq i \leq n$. Each A_i possesses an identity and is simple as a ring. And $A \simeq \prod_{i=1}^n A_i$. Conversely, a finite direct product of simple rings B_i is semisimple with B_i being the minimal two-sided ideals.*

Proof. The A_i are the isotypic components, so $A = \oplus_{i=1}^n A_i$. We have $A_i A_j \subseteq A_i \cap A_j = 0$. The component in A_i of the identity element of A is an identity for the multiplication on A_i. It is easy to check that $A \simeq \prod_{i=1}^n A_i$. □

The A_i as in the theorem are called the *simple components* of the semisimple ring A.

- Any quotient of A is semisimple and isomorphic to a product of some of the simple components A_i.
- The number of isomorphism classes of simple A-modules equals the number of simple components of A. For a simple A-module M, there is a unique simple component A_i such that $A_i M \neq 0$. Considered as an A_i-module, M is simple.

[ss:dhi.tex]

9.4. **Degrees, heights, and indices.**

[p:dhi:1]

Proposition 9.4. *Let B be a simple ring of length r and M a B-module.*

(1) *M is free if and only if $\ell_B M$ is either a finite multiple of r or infinite.*
(2) *If M is free then $\ell_B M = r \cdot |\mathfrak{B}|$ (where $|\mathfrak{B}|$ denotes the cardinality of a B-basis for M).*
(3) *If M is a B-B-bimodule, then M is free.*

Proof. If $\ell_B M$ is either a finite multiple of r or infinite, then M is isomorphic to a direct sum of ${}_B B$ (transfinite induction?). Conversely, if M is free then $M = ({}_B B)^{\oplus |\mathfrak{B}|}$, and so $\ell_B M = r \cdot |\mathfrak{B}|$. This proves (1) and (2). Finally, if M is a bimodule, then $M \simeq M \otimes_B B \simeq M \otimes_B (S^{\oplus r}) \simeq \oplus_r (M \otimes_B S)$, where S is the simple B-module. So $\ell_B M$ is a multiple of r and M is free. □

[sss:degree]

[$A^{\mathrm{opp}} : B^{\mathrm{opp}}$] could be quite different from $[A : B]$; in fact, one could be finite and the other infinite, even when B is a field.

9.4.1. Degree. Now let A be a ring and B a simple subring (containing the identity). Then A is free as a (left) B-module and the cardinality of any two bases agree, for A is a bimodule. The *B-degree of A*, denoted $[A : B]$ is defined to be the cardinality of a B-base for A.

- When B is a divsion ring, $[A : B]$ is just the vector space dimension of A.
- For C a simple subring of B (containing the unit), $[A : C] = [A : B][B : C]$.
- For a B-module M, we have $\ell_B(A \otimes_B M) = [A : B]\, \ell_B(M)$. (Proof: Let S be the simple B-module and $r := \ell(B)$. On the one hand, $\ell\,(A \otimes_B M) = \ell\,(A \otimes_B (S^{\oplus\, \ell(M)})) = \ell\,(A \otimes_B S)\, \ell(M)$. On the other, $\ell\,(A \otimes_B S) \cdot r = \ell\,(A \otimes_B (S^{\oplus r})) = \ell\,(A \otimes_B B) = \ell\,(A)$. And $\ell\,(A) = [A : B] r$ by the proposition.)

[sss:hi]

9.4.2. Height and index. Let now A be a simple ring and B a simple subring (containing the unit). Let S and T denote the respective simple modules. The *height*, denoted $\mathfrak{h}(A, B)$, is $\ell_B(S|_B)$; the *index*, denoted $\mathfrak{i}(A, B)$, is $\ell_A\,(A \otimes_B T)$.

We have:

(1) $\ell_A(A) = \mathfrak{i}(A, B)\, \ell_B(B)$ (Proof: $\ell_A(A) = \ell_A(A \otimes_B B) = \ell_A(A \otimes_B T^{\oplus\, \ell_B(B)}) = \ell_A(A \otimes_B T)\, \ell_B(B) = \mathfrak{i}(A, B)\, \ell_B(B)$.)

(2) For any A-module P, we have $\ell_B(P) = \ell_A(P)\mathfrak{h}(A,B)$ (for $P = S^{\oplus \ell_A(P)}$).
(3) If A and B are finite dimensional over an algebraically closed field k, then $\mathfrak{i}(A,B) = \mathfrak{h}(A,B)$ (by Frobenius reciprocity).
(4) Using the proposition above, we get $[A : B] = \mathfrak{i}(A,B)\mathfrak{h}(A,B)$ (Proof: We have $[A:B] = [A:B]\,\ell_B(T) = \ell_B(A \otimes_B T) = \ell_B(S^{\oplus \mathfrak{i}(A,B)}) = \mathfrak{i}(A,B)\mathfrak{h}(A,B)$.)
(5) If $C \subseteq B$ with C simple and containing the unit element, then $\mathfrak{i}(A,C) = \mathfrak{i}(A,B)\mathfrak{i}(B,C)$ and $\mathfrak{h}(A,C) = \mathfrak{h}(A,B)\mathfrak{h}(B,C)$.

[p:aba'b']

Proposition 9.5. *Furthermore, let M be a non-zero A-module such that $\ell_B(M)$ is finite; let A' and B' denote respectively the commutants of M as an A-module and B-module. Then A' and B' are simple rings and A' is a subring of B' containing the unit element. We have:*

$$\mathfrak{i}(B',A') = \mathfrak{h}(A,B) \qquad \mathfrak{h}(B',A') = \mathfrak{i}(A,B) \qquad [B' : A'] = [A : B]$$

Proof. By item (2) above, $\ell_B(M) = \ell_A(M)\mathfrak{h}(A,B)$, so both factors on the right are finite. We have already seen that the commutant of a semisimple isotypic module of finite length r is simple of length r. So A' and B' are simple rings of lengths $\ell_A(M)$ and $\ell_B(M)$ (Corollary **??**). Combining this with item (1) above gives $\ell_B(M) = \mathfrak{i}(B',A')\,\ell_A(M)$. Comparing with the equation deduced earlier from item (2), we conclude that $\mathfrak{h}(A,B) = \mathfrak{i}(B',A')$.

From Corollary **??**, it follows that A' and B' are their own bicommutants. We can therefore switch the roles of A and B with B' and A' to conclude that $\mathfrak{i}(A,B) = \mathfrak{h}(B',A')$.

Finally, the last equation follows now from item (4) above. □

[ss:xsssrings]

9.5. **Exercises.**

9.5.1. A. ring is *primitive* if it has a faithful simple module. It is possible that primitive rings have non-trivial proper two sided ideals (e.g., the ideal of finite rank linear transformations in the full ring of linear transformations of an infinite dimensional vector space). Prove, however, that an Artinian primitive ring is simple.