LECTURE 4.1 — ROUQUIER COMPLEXES

Rouquier Complexes are the Soergel Bim incarnation of many well-known constructions in other contexts – (twisting+)shuffling functors, spherical functors, etc, that give braid gp actions.

We've seen two SES of R-bim

$$0 \to R(-1) \to B_s \to R^s(1) \to 0$$
$$0 \to R_s(-1) \to B_s \to R(1) \to 0$$

which yield

$$0 \to \overset{-1}{R(-1)} \to \overset{0}{B_s} \to 0 = F_s^{-1}$$
$$\downarrow \qquad \downarrow$$
$$0 \to R_s(-1) \to 0$$

qisom, no inverse map, not h.e.

Sim. $0 \to \overset{0}{B_s} \to \overset{1}{R(1)} \to 0 = F_s$

$$\uparrow$$
$$0 \to R_s(-1) \to 0$$

In the usual euler characteristic map, $[F_s^{-1}] = [B_s] - [R(-1)] = \underline{H}_s - v = H_s$

$[F_s] = \underline{H}_s - v^{-1} = H_s^{-1}$.

F_s is more useful, whence our -1 convention

Def: Let $K^b(\mathbb{S}Bim)$ denote the homotopy cat of $\mathbb{S}Bim$ (can do this for any additive cat)

Ob: Bounded cxs of $\mathbb{S}Bim$ (degree 0 differentials) Mor: Chain maps modulo homotopy.

Let $D^b(RBim)$ be the derived cat of R-Bim (can only do this for abelian cat)

Add inverses of qisoms.

Def: Rouquier Complexes are $F_{\underline{w}} = F_s \otimes F_t \otimes \dots \otimes F_u \in K^b(\mathbb{S}Bim)$ (inverses too)

Ex: $F_s \otimes F_s^{-1}$

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & & B_s \otimes B_s & & \\ B_s(-1) & & \oplus & & B_s(+1) \\ & & R & & \end{array}$$

$$[\ ] \uparrow \downarrow \qquad [\ ]$$

$$0 \longrightarrow R \longrightarrow 0$$

(a) shifts = hom degree (differentials "deg 1")
(b) all maps are single dots
(c) sign is $(-1)^{\#\text{ lines before it}}$ (we'll use another)

These maps give you a hom. eq.

$F_s \otimes F_s^{-1} = \mathbb{1}$ monoidal identity

Ex: $F_s \otimes F_s$

Exercise

$$B_s \otimes B_s \to B_s(1) \oplus B_s(1) \to R(2) \quad \simeq \quad B_s(-1) \to B_s(1) \to R(2)$$

$B_s \otimes B_s = B_s(-1) \oplus B_s(1)$

qisom to R but not h.e. !!

bad shift too.

Ex: $m_{s,t} = 3$

$F_s \otimes F_t \otimes F_s$

$$B_sB_tB_s \to B_sB_t(1) \oplus B_sB_s(1) \oplus B_tB_s(1) \to B_s(2) \oplus B_t(2) \oplus B_s(2) \to R(3)$$

$B_sB_s(1) = B_s \oplus B_s(2)$

$B_sB_tB_s = B_{sts} \oplus \cancel{B_s}$

$$\simeq \quad B_{sts} \to B_sB_t(1) \oplus B_tB_s(1) \to B_s(2) \oplus B_t(2) \to R(3)$$

KEEP EXAMPLES ON BOARD

Thm (Rouquier): $F_s^{\circ}, F_s^{-1}{}^{\circ}$ give a <u>strict</u> categorification of the braid gp of W in $K(\mathbb{S}Bim)$ ②

I.e. F_s satisfy braid relations up to h.e., F_s, F_s^{-1} are inverse functors

and $\mathrm{End}(F_{\underline{w}}) = R$! However, faithfulness is still an open problem!
i.e. $F_{\underline{w}} \cong F_{\underline{y}} \Rightarrow \underline{w} \neq \underline{y}$ in braid gp
(only known in types ADE)

Also, they give a strict faithful categorification of W on $D^b(R\text{-}Bim)$
since $F_s \cong 0 \to R_s(-1) \to 0$ $\quad F_s^{-1} \cong 0 \to R_s(1) \to 0$ $\quad F_sF_s^{-1} \cong 0 \to R \to 0$.
$\Rightarrow$ know cohomology.

<u>Rmk</u>: (E-Krasner) For you topological folks - any braid cobordism gives chain map, get action of BrCob.

Let's look at the examples we've seen. Whenever $B_x(n)$ appeared in two adjacent degrees, there was a homotopy contracting the two summands away. What's with that?

<u>Fun Homological Alg</u>: Let A be a (graded) local ring. Then inside $K^b(A\text{-mod})$, any complex C° is h.e. to a <u>minimal</u> complex C°_{min}, for which all differentials lie in the maximal ideal $\Leftrightarrow$ no contractible summands! Any two such are (non-canonically) isomorphic. Why? Any differential not in the max ideal gives an isom b/w two summands, can contract it.

<u>Exercise</u>: $\mathrm{End}(\oplus B_w)$ is a graded local ring. Modify the above to deduce that minimal complexes exist in $K^b(\mathbb{S}Bim)$. Let $F_w \equiv F_{\underline{w},min}$ for any red exp, only F_s, no F_s^{-1}. (positive braids)

Examples you've seen.
However, we can't deduce that adjacent B_x's can be eliminated, since we don't know that $\mathrm{End}(B_x)^0 = \mathbb{R}$, there might be deg 0 maps in maxl ideal. If SConj holds, any nonzero diff $B_x(n) \to B_x(n)$ can be cancelled.

<u>Ex</u>: F_{tsut} in S_4.

(0)	(1)	(2)	(3)	(4)
$B_tB_sB_uB_t = B_{tsut}$	B_{tsu}	B_{ts}	$\cancel{B_t}$	R
	B_{tst}	B_{tu}	B_s	
	$\cancel{B_t}$	B_{st}	B_u	
	B_{tut}	B_{su}	B_t	
	B_t	B_{ut}		
	B_{sut}	$\cancel{B_t(1)}$		
		$\cancel{B_t(-1)}$		

they're not so obvious.

don't know maps so well but know size.

Now for the key properties of Rouquier complexes: ③

~~Exercise: not finished~~ Def: $K^{\leq 0}$ = Complexes h.e. to those where degree i has all shifts $\geq i$

$K^{\geq 0}$ = ″ shifts $\leq i$ (SConj $\Rightarrow$ t-structure)

Ex: Most of what we've seen is in the core $K^{\leq 0} \cap K^{\geq 0}$

But $F_s F_s \in K^{\geq 0} \cap K^{\leq 1}$ $F_s^{-1} F_s^{-1} \in K^{\leq 0} \cap K^{\geq -1}$.

Exercise: $\underline{\omega}$ a positive braid, then $F_{\underline{\omega}} \in K^{\geq 0}$ (shifts are $\leq i$ what they should be)

Hint: Show that whenever $B_s \otimes \cdot$ makes the shift go up, it is cancelled by $\to R(1)$.

Should assume SConj for this exercise - that way $B_s B_x \cong \bigoplus B_y^{\mu(y,sx)}$ w/ no shifts

Thm (Diagonal Miracle): $F_w \in K^{\leq 0} \cap K^{\geq 0}$, it is B_w in degree 0.

$\boxed{\text{Assume } S(y)\ \forall y \leq w}$

Assuming this, we get nice "formulas" for inverse KL polynomials! Go back to F_{w_0} and count the appearances of $B_{\bar{f}}$, for instance. Gives formula for $H^{-1}_{\text{w}_0}$.

The proof uses ~~[illegible]~~ std filtrations and a result of W-Libedinsky stating that Rouquier complexes split on the associated graded. Slightly technical. We won't truly need ~~it~~ to prove SConj, but it helps speed things up.

Homology: $H^*(F_{\underline{\omega}}) = H^*(F_w) = R_w(-l(w))$ in cohom degree 0, nothing else.

i.e. free R-mod generated in degree $l(w)$

Thus the map $B_w \hookrightarrow BS(\underline{\omega}) \xrightarrow{\Sigma\,\text{ill pill}} \bigoplus BS(\hat{\underline{\omega}}_i)$ is injective below degree $l(w)$

ignore signs, we're different sign convention

What could we possibly use that for? ...