Ben Elias 1/2014 Chennai Colloquium 2 Q'm Geo Satake w/o Tears ①

Fast intro: GS is a very powerful + lovely thm, connecting geom. associated to affine Weyl gp to reps thry of alg Lie gp. Hard enough to state, unintuitive + very technical proof via Tannakian Formalism + relying on Decomposition Theorem. Tears!

I will state + prove (almost) an equiv of 2-cats which is easier in every way, and equivalent. Only in type A tho. Then I'll quantize the theorem. First examples, then statement, then more intro last.

Rep Thry of $\mathfrak{sl}_2$ | $\mathrm{Rep}_{\mathfrak{sl}_2}$ is s.s. monoidal cat/$\mathbb{C}$: ireps V_n $n \in \mathbb{N}$

$\otimes$ with $V_1 \otimes V_n \cong V_{n+1} \oplus V_{n-1}$
$V_1 \otimes V_0 \cong V_1$

Semisimple cats are easy... but not as monoidal cats.
i.e. $\mathrm{Hom}(V_n, V_m) = \delta_{n,m} \cdot \mathbb{C}$ is easy, $\mathrm{Hom}(V,W)$ is easy, but:

Q1: Describe $\bigoplus_{\underline{n},\underline{m}} \mathrm{Hom}(V_{n_1} \otimes V_{n_2} \otimes \dots \otimes V_{n_d}, V_{m_1} \otimes \dots \otimes V_{m_e})$ w/ its two compositions - vertical + horizontal.

There is an answer, but not an easy one - for $\mathfrak{sl}_3$ even, no answer! $\mathrm{Rep}_{\mathfrak{g}}$ is hard!

General philosophy - when studying difficult monoidal cats, find a nice monoidal subcat.

$\mathrm{Fund}_{\mathfrak{sl}_2}$ has objects $V_1^{\otimes n} \leftrightarrow \mathbb{N}$. Clearly by recursive formula, $V_n \overset{\oplus}{\subset} V_1^{\otimes n}$, so

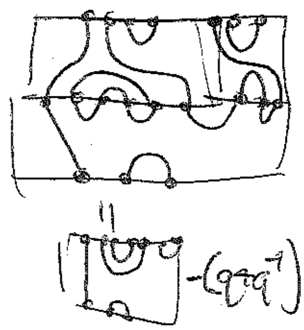
$\mathrm{Kar}(\mathrm{Fund}_{\mathfrak{sl}_2}) = \mathrm{Rep}_{\mathfrak{sl}_2}$. This leads to
a process which adds summands.

Q2: Describe $\bigoplus_{n,m} \mathrm{Hom}(V_1^{\otimes n}, V_1^{\otimes m})$ w/ two compositions. This has a famous + elegant answer.

Def: Let TL_q be monoidal cat w/ $\mathrm{Ob} \leftrightarrow \mathbb{N}$ $n \otimes m = n+m$. (think $V_1^{\otimes n}$)
generated by $\underline{1}$.

$\mathrm{Hom}(n,m) = \mathbb{Z}[q,q^{-1}] \langle (n,m)\text{-crossingless matchings} \rangle$

horiz comp = concat, vert comp = concat + resolve circles
$\bigcirc = -(q+q^{-1})$



Old Thm: There is a functor $TL/_{q=1} \xrightarrow{\sim} \mathrm{Fund}_{sl_2}$
(Kauffman?)
(Jones?)

$n \longmapsto V_1^{\otimes n}$

$V_0 \cong \Lambda^2 V_1$, incl, proj. $V_1^{\otimes 2}$

 $\longmapsto$ certain morphisms

can choose scalars correctly so $\boxed{\ominus} = -2$

Rmk: So Q2 solved. To return to Q1, can find description of $e_n \in \mathrm{End}(V_1^{\otimes n})$ projecting to V_n. Jones-Wenzl projector. However, analogous idempotents not known beyond rank 2.

Key remark: $\mathrm{Rep}_{sl_2} \circlearrowright \mathrm{Rep}_{sl_2}$ (like regular reps)
$\| $
$\mathrm{Even}_{sl_2} \oplus \mathrm{Odd}_{sl_2}$ so they acting should resemble 2×2 matrices.

I.e even tho V_1 is irred, $V_1 \otimes (\cdot)$ decomposes into $\left\{ \begin{matrix} W_{even} \mapsto (W \otimes V_1)_{odd} \\ W_{odd} \mapsto 0 \end{matrix} \right\} \oplus \left\{ \begin{matrix} \text{vice} \\ \text{versa.} \end{matrix} \right\}$

Correct way to encode this structure is in a 2-cat

Here's the Fund_{sl_2}-version: $2TL \cong \mathrm{Fund}_{sl_2}^{\mathbb{Z}/2\mathbb{Z}}$

Obj: even, odd (red, blue)
1-mor: gen by even → odd and odd → even
2-mor: Colored crossingless matchings

This is more natural than TL!!

Singular Soergel Bimodules for $\widehat{sl_2}$ Let h^* be refl rep of $W_{aff}(sl_2) = \langle s,t \mid s^2=t^2=1 \rangle$
$\hat{W}$

i.e $h^* = \mathbb{C}\langle \alpha_s, \alpha_t \rangle$ $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ $s(\alpha_t) = \alpha_t + \alpha_s$ $s(\alpha_s) = \alpha_s - \alpha_s = -\alpha_s$

Let $R = \mathrm{Sym}(h^*) = \mathbb{C}[\alpha_s, \alpha_t] \circlearrowleft W$, graded w/ $\deg \alpha_s = 2$

$R^s = \{ f \mid sf = f \} = \mathbb{C}[\alpha_s^2, \alpha_s + 2\alpha_t] \subset R$ is a finite ext by Chevalley's Inv't Thm

Even better, a Frob. Ext, essentially Ind + Res are biadjoint (Ind $\cong$ Coind) up to grading shift

Part of structure: $\partial_s : R \to R^s$
$f \longmapsto \frac{f - s(f)}{\alpha_s}$.

Note! $R^W = \mathbb{C}[\alpha_s + \alpha_t] \subset R$ is NOT nice, b/c W is infinite.

Consider ${}_{R^s}R_{R}(1)$ and ${}_{R}R_{R^s}(1)$ bimodules. $\otimes$ gives functors $R^s\text{-mod} \rightleftarrows R\text{-mod}$
Res ∘ Ind

1 in degree −1.

Def: mSBSBim is 2-cat w/ Ob: s, t

1-mor: gen by ${}_{s}R_{t}(1)$ ${}_{t}R_{s}(1)$

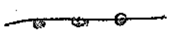
2-mor: graded bimodule maps

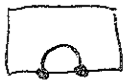
$\| \; \mathcal{C}$

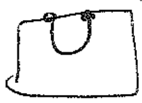
Rmk: This, like Fund$_{sl_2}$, is a nicer replacement. Kar(mSBSBim) = mSSBim is something natural, as we'll see. Same philosophy. Also has comb. description.

Thm: There is a 2-functor $\mathcal{F}: \mathcal{Q}TL \xrightarrow{\sim_0} \mathcal{C}$

even ↦ s, odd ↦ t (colors)

 ↦ ${}_{R^t}R \otimes_{R^s} R \otimes_{R^t} R_{R^s}$

 ↦ $R^t \longleftarrow R \otimes_{R^s} R$, $\quad \partial_t(fg) \longleftarrow f \otimes g$

 ↦ $R \otimes_{R^t} R \longleftarrow R^t$, $\quad \frac{1}{2}(\alpha_t \otimes 1 + 1 \otimes \alpha_t) \longleftarrow 1$ (Casimir)

Rmk: $\mathcal{F}$ not really an equiv — Hom spaces in $\mathcal{C}$ are graded vect, image of $\mathcal{F}$ is degree 0. No, mean: Homs b/w 1-mor in image of $\mathcal{F}$ are in nonneg degrees, $\mathcal{F}$ fully faithful to degree 0.

Rmk: $\mathcal{C}$ has higher degree morphisms. Ex: $\mathrm{End}({}_{R^t}R_{R^t}(1)) = R$ + ∃ comb description of those too (even easier than just doing degree 0)

Alg Satake: $\mathrm{Fund}_{sl_2}^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\sim_0}$ mSBSBim

$\Rightarrow \mathrm{Rep}_{sl_2}^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\sim_0}$ mSSBim

Nothing has tears.

General Statement: Fix $\mathfrak{g}$. $\mathrm{Fund}_{\mathfrak{g}} \subset \mathrm{Rep}_{\mathfrak{g}}$ is $\otimes$ of fund. reps.

$\mathrm{Rep}_{\mathfrak{g}}$ is Ω-graded $\Omega = \Lambda_{wt}/\Lambda_{rt}$. $\mathrm{Fund}_{\mathfrak{g}}^{\Omega}$, $\mathrm{Rep}_{\mathfrak{g}}^{\Omega}$ the 2-category.

$\mathfrak{h}^*$ refl rep of W_{aff}, simple refl $\leftrightarrow \tilde{\Gamma}$  For $I \subsetneq \tilde{\Gamma}$, $W_I \subset W_{aff}$ is finite.

$R = \mathrm{Sym}(\mathfrak{h}^*)$ $\quad R^I = R^{W_I}$. $\quad$ SSBim = Ob: $I \subsetneq \tilde{\Gamma}$

1-mor: (R^I, R^J) bimodules which are summands of compositions of Ind, Res + ⊕ + (1)

2-mor: bimodule maps.

$_M\mathrm{SBBim}$ = part where $I = \tilde{\Gamma} \setminus$ removable vertex (i.e. $I \cong \Gamma$)
in bij w/ Ω.

~~Alg~~ Soergel Satake: $\mathrm{Rep}_q^{\Omega} \xrightarrow{\sim} {}_M\mathrm{SBBim}$.

How to prove ??? In type A, find nice subcats.

Fund_q^{Ω} has nice description via "$^q\mathfrak{sl}_n$-webs" Cautis–Kamnitzer–Licata 2013
rank 2: Kuperberg

$_M\mathrm{SBBim}$ has nice description via "singular Soergel diagrams" Elias–Williamson 2015.

gen by ~~...~~ $\mathrm{Res}^{R^{\tilde\Gamma\setminus a,b}}_{R^{\tilde\Gamma\setminus b}} \circ \mathrm{Ind}^{R^{\tilde\Gamma\setminus a,b}}_{R^{\tilde\Gamma\setminus a}}$

directly show equivalence!

Alg. Satake.

To quantise: $\mathrm{Rep}_{U_q(\mathfrak{g})} \supset \mathrm{Fund}_{U_q(\mathfrak{g})}$ also has description (this is not setting $q=1$)

Using q-deformed Cartan matrix, get $_M\mathrm{SBBim}_q$, equiv still works!!!

$$\begin{pmatrix} 2 & -q-q^{-1} \\ -q-q^{-1} & 2 \end{pmatrix} \qquad \left(\begin{array}{ccccc|c} 2 & -1 & & & & -q^{-1} \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & & -q \\ \hline -q & & & -q^{-1} & & 2 \end{array}\right)$$

← No explanation whatsoever geometric or otherwise. A great mystery!

Time for tears. What does Soergel Satake have to do with Geom. Satake

GS: G, $(G^\vee)$ a (Langlands dual) Lie gp $\quad K = \mathbb{C}((t)) \quad G^\vee(K)$
$\mathcal{O} = \mathbb{C}[[t]] \quad G^\vee(\mathcal{O})$

Exs: GL_n GL_n
SL_n PGL_n

$$\mathrm{Perv}_{(G^\vee(\mathcal{O}),G^\vee(\mathcal{O}))}(G^\vee(K)) \xrightarrow{\sim} \mathrm{Rep}\, G$$
H^* ↘ Vect ↙ forget

Rmk: $\mathrm{Perv}_{(G^\vee(\mathcal{O}),G^\vee(\mathcal{O}))}(G^\vee(K)) \cong \mathrm{Perv}_{G^\vee(\mathcal{O})}(G^\vee(K)/G^\vee(\mathcal{O}))$ but monoidal structure is only natural on former
affine Grassmannian

Transform to Soergel S in 3 easy steps

Step 1: Replace w/ 2-cat. $G^\vee(K)$ is typically disconnected so graded by connected components – $\pi_0(G^\vee(K)) = \Omega$ (or some subgp...)

Natural way to unravel this: $\mathrm{Perv}_H(xG^\vee(K)^{\mathrm{nul}}) \cong \mathrm{Perv}_{xHx^{-1}}(G^\vee(K)^{\mathrm{nul}})$ ← mostly indep of choice of $G^\vee$, after $/G^\vee(\mathcal{O})$

So instead work with 2cat: Ob: $X \in \Omega$

$\mathrm{Hom}(1, X) = \mathrm{Perv}_{(G^\vee_1, G^\vee_X)}\left(G^\vee(K)^{red}\right)$

where $G^\vee_X = X G^\vee(\mathcal{O}) X^{-1}$ Rmk: Actually makes sense for arbitrary $G^\vee$, $X \notin G^\vee(K)$ but X acts by outer automorphism

Ex: $G = GL_n$ or SL_n or PGL_n $\quad X = \begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\quad G(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & & \\ & \mathcal{O} & \\ & & \mathcal{O} \end{pmatrix}$

$G_X = \left(\begin{array}{c|c} \mathcal{O} & t\mathcal{O} \\ \hline t^{-1}\mathcal{O} & \mathcal{O} \end{array}\right)$ makes sense in $SL_n(K)$ even though $X \notin SL_n(K)$

Step 2: Replace $G^\vee(K)$ etc w/ Kac-Moody groups

To $\tilde{\Gamma} \supset I$ have $G_{\tilde\Gamma} \supset P_I$ and $G_{\tilde\Gamma} / P_{\tilde{\Gamma}_{fin}} \cong G^\vee(K) / G^\vee(\mathcal{O})$

so work w/ $\mathrm{Perv}_{(P_I, P_J)}(G_{\tilde\Gamma})$ Advantage – space is same but group action is bigger, more equivariant structure!!

Step 3: Take equiv. global sections Soergel-Harterich: this is fully faithful on semisimple perverse sheaves, showed image was SBBim

Get bimodules over $\left(H^*_{P_I}(pt), H^*_{P_J}(pt)\right)$. Now ~~$H^*(pt)$~~ $H^*_{P_I}(pt) \cong R^I$ inside $H^*_B(pt) = R$

(If hadn't done step 2, would get ~~$H^*(pt) = \ldots$~~ nothing exciting. $H^*_{G(\mathcal{O})}(pt) = R_{fin}^{W_{fin}}$ inside $H^*_I(pt) = R_{fin}$)

Step 4: Find comb subcats, to make life easy! A key aspect of Soergel's approach in general.