LECTURE 3.4: LIGHTNING INTRO TO PERVERSE SHEAVES

Goal of this lecture – $\exists$ 2 descriptions of B_w, when Soergel's conjecture holds.

1. $ch(B_w) = b_w$, i.e. $ch(B_w)$ is self-dual, $ch(B_w) = v^{\ell(w)}(T_w + \sum P_{yw} T_y)$
   w/ degree restrictions

2. $B_w \overset{\oplus}{\underset{1}{\subseteq}} B_{s_1} \otimes \ldots \otimes B_{s_d}$, etc.

We want to motivate why Soergel thought these should be the same.

$\exists$ some abelian category of perverse sheaves on ${}_B\backslash G/B$, with simples IC_w, $w \in W$.

We'll discuss why IC_w has 2 descriptions, analogous to the above.

There is a functor $\mathrm{Perv}_{B\times B}(G) \xrightarrow{\Gamma_{B\times B}} R\text{-Bimod}$, and SBim is the image of the semisimple objects. Thus $\Gamma_{B\times B}(IC_w)$ has 2 descriptions.

Much will be left to the exercises.

There are some topics for which even 45 minutes can't make you an expert! The amazing thing is that, despite being about as difficult + technical as it gets, you can learn enough to do computation w/ perverse sheaves quickly!

Start w/ a stratified space: Λ a poset, $X = \coprod_{\lambda \in \Lambda} X_\lambda$ smooth of $\mathbb{C}$-dim $d(\lambda)$, $\overline{X_\lambda} = \coprod_{\mu \leq \lambda} X_\mu$
(ignore technical conditions) can be singular

Ex: $G \curvearrowright X$, stratified by orbits.

Ex 1: $B \curvearrowright G/B = \coprod_{w \in W} BwB/B$

$\overline{BwB}/B$ a Schubert variety

$BwB/B \cong \mathbb{C}^{\ell(w)}$

Ex 2: $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_n$

$P \curvearrowright Gr(k,n)$

$P = \mathrm{Stab}(0 \subset V^{d} \subset V^{d_2} \subset \ldots \subset V^n)$

Orbits on $\{W^k \subset V^n\}$ determined by numbers $\dim(W^k \cap V^{d_i})$

numbers increase in closure.

Often interested in proper stratified resolutions of singularities of $\overline{X_\lambda}$, i.e. $Y \xrightarrow{f} \overline{X_\lambda}$

Y smooth, s.t. $f^{-1}(X_\mu) \to X_\mu$ is a fiber bundle w/ compact fiber F_μ, and $F_\lambda = *$.

Ex 1: $\underline{w} = s_1 \ldots s_d$ red exp. $P_{s_i}/B \cong \mathbb{P}^1$

$Y = P_{s_1} \overset{B}{\times} P_{s_2} \overset{B}{\times} \ldots \overset{B}{\times} P_{s_d}/B \xrightarrow{mult} G/B$

$(p, p') = (pb, b^{-1}p')$ Bott-Samelson Resolution, iterated $\mathbb{P}^1$-bundle

$Y = \{W^{k_1} \subset W^{k_2} \ldots \subset W^{k} \subset V\}$

fix the intersections themselves of $\{W^k \subset V\}$

For rest of today - assume $X_\lambda \cong \mathbb{C}^{d(\lambda)}$ (or at least $\pi_1(X_\lambda) = 1$) ②

A constructible sheaf on X is ... I won't tell you. But to F const, $x \in X$ can take stalk F_x a f.d. v.s., only depends on stratum $x \in X_\lambda$, so call it F_λ. Sheaf $\rightsquigarrow$ Table of natural numbers

λ	$F_\lambda = \mathbb{C}^{n_\lambda}$
μ	$\mathbb{C}^{n_\mu}$
ν	$\mathbb{C}^{n_\nu}$

TABLE DOES NOT DETERMINE SHEAF

The constructible derived category $D(X)$ is ... to $F^\bullet \in D(X)$, can take cohomology sheaves $H^i(F^\bullet)$ for $i \in \mathbb{Z}$, each has a table, so $F^\bullet \rightsquigarrow$ Table

	-2	-1	0	1	2
λ	0	$\mathbb{C}^1$	0	0	
μ	0	$\mathbb{C}^n$	$\mathbb{C}^p$		
ν			etc.		

TABLE DOES NOT DETERMINE SHEAF

Ex: ⓐ $F = \mathbb{C}_X$ ⓑ $F = \mathbb{C}_{\overline{X_\lambda}}$ ⓒ $F = \mathbb{C}_X[2]$

ⓓ $f_* \mathbb{C}_Y$ has on λ^{th} row, $H^*(F_\lambda)$

Ex 2: $P = \begin{pmatrix} * & * \\ & * \end{pmatrix} \subset GL_4$ (blocks of size 2, 2), $0 \subset V^2 \subset \mathbb{C}^4$

$Y = \{ L^1 \subset W^2 \subset \mathbb{C}^4, L^1 \subset V^2 \}$

$\big\downarrow$ forget L

$X_1 = Gr(2,4)_1$

$\overline{X_1} = \{ W^2 \subset \mathbb{C}^4 : \dim W \cap V \geq 1 \}$

ⓔ $f_* \mathbb{C}_Y[3]$

	-3	-2	-1	0	
$Gr(2,4)_0$	0	0	0	0	fiber $= \emptyset$
$Gr(2,4)_1$	$\mathbb{C}$	0	0	0	$= *$
$Gr(2,4)_2$	$\mathbb{C}$	0	$\mathbb{C}$	0	$= \mathbb{P}^1$

Ex 1: In type A_2, $P_s \times^B P_t \times^B P_s / B \xrightarrow{\mu} G/B$

$\mu_* \mathbb{C}[3]$

	-3	-2	-1	0	fiber
sts	$\mathbb{C}$				$*$
st	$\mathbb{C}$				$*$
ts	$\mathbb{C}$				$*$
t	$\mathbb{C}$				$*$
s	$\mathbb{C}$		$\mathbb{C}$		$\mathbb{P}^1$
1	$\mathbb{C}$		$\mathbb{C}$		$\mathbb{P}^1$

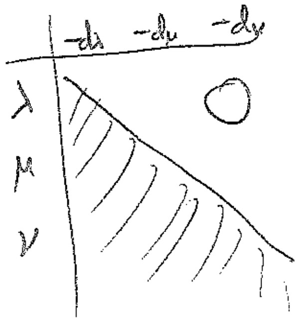
Compare this with $b_s b_t b_s = v^3 \sum_{x \leq sts} T_x + v^1 \sum_{x \leq s} T_x$

encode table for G/B using std basis of $\mathbb{H}$.

get $ch(F^\bullet)$

Def: $F^\bullet \in D(X)$ is perverse if

a)



now draw lines above

b) Same condition for the Poincaré dual $\mathbb{D}(F^\bullet)$ complicated + mysterious. Though if $F^\bullet \cong \mathbb{D}(F^\bullet)$, ⓐ suffices.

Key facts: ① If X smooth, $\mathbb{D}\mathbb{C}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$, so $\mathbb{C}_X[\dim X]$ is self-dual.

② If f proper, $\mathbb{D} f_* = f_* \mathbb{D}$ so f_* preserves self-duality

$\Rightarrow$ examples above are self dual $\Rightarrow$ perverse.

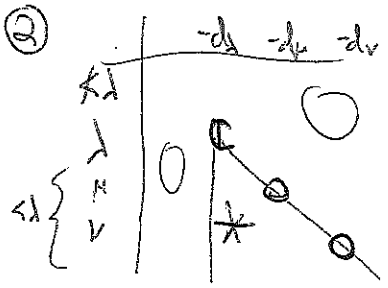
Thm: Perv is abelian. Simples $\longleftrightarrow \Lambda$

$IC_\lambda \longleftrightarrow \lambda$

(This was exactly condition for $ch(IC_\lambda) = b_\lambda$.)
in G/B case.

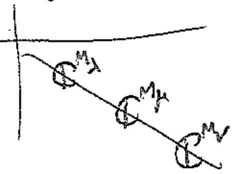
IC_λ !-ly specified by ① self-dual

②



Ex 2 is IC. Ex 1 is NOT, but is $IC_{sts} \oplus IC_s$
(or maybe a weird extension?)

If $\mathcal{F}^\circ = \bigoplus IC_\lambda^{\oplus m_\lambda}$, $m_\lambda \in \mathbb{N}$ then self-dual,



$\rightsquigarrow m_\lambda \in \mathbb{N}[v+v^{-1}]$

semisimple in $D(X)$

TABLE DETERMINES SHEAF !!!

Decomposition Thm: $Y \xrightarrow{f} X$ proper, then f_* preserves semisimples

Cor: $Y \xrightarrow{f} X$ res of sing of $\overline{X_\lambda}$ then $IC_\lambda \overset{\oplus}{\underset{1}{\subseteq}} f_* \underline{\mathbb{C}}_Y[\dim Y]$, other summands are $IC_\mu^{\oplus m_\mu}$, $\mu < \lambda$, $m_\mu \in \mathbb{N}[v+v^{-1}]$

Rmk: Can use this fact to inductively compute $ch(IC_\lambda)$, just as we inductively compute b_w.

i.e. find resn, take $f_* \underline{\mathbb{C}}_Y[\dim Y]$, cross off lower terms, what remains is IC_λ.

See example 1.

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aside?
So we have a (table-theoretic / combinatorial) description of $s.s. \subset D(X)$. What can you do with non-semisimple perverse sheaves? Best approach — understand in terms of extension maps b/w simples

(Koszul dual side is more familiar — to understand a non-projective module, take projective resolution)

As $SBim = \Gamma^{*}_{B\times B}(s.s.)$, we'll understand non-s.s. sheaves using complexes of SBim. Rouquier complexes, more to come.

Can you guess $\Gamma^{*}_{B\times B}(\mathcal{F}^\circ)$ from the table? Not easily. But $\Gamma^{*}_{B\times B}(\underline{\mathbb{C}}_Y) = \Gamma^{*}_{B\times B}(f_* \underline{\mathbb{C}}_Y)$ is easy, which is why we work with Bott-Samelsons. Exercises.

Exercises for 3.2 - Sketch

Aside about pullbacks.

1. Define sl_2 structure on $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset Gr(k,4)$, Compute all IC_s, and what E, F do to each
   One key comput. $H^*(E, IC_s)$ on $Gr(2,4)$ Guide thru. Filtration $V_4 \oplus V_2 \oplus V_0$, certify $V_2 \otimes V_2$
   Try more examples!

2. Compute fibers in a bunch of BS Resolutions for S_4. ss; Ext; switch; split into IC_s

3. (Formal stuff) Define convolution. Check that $\mu_*(BS(\underline{w})) = IC_{s_1} * IC_{s_2} * \dots * IC_{s_d}$

4. Calculations w/ equiv. cohom. in type A. $H^*(Fl(0,1,\dots,n,\infty))$, $Fl(0,n,\infty)$ etc.
   Ind, Res,
   Calculate for BS resolution, get $BS(\underline{w})$

5. ~~[illegible]~~ Define stds $T_w = \underline{\mathbb{C}}_X$. Check convolution ~~[illegible]~~
   Check $IC_s * \dots$ satisfies rel equations
   6.