BEN ELIAS UNC Colloquium 9/2013 Hopfological Algebra ①

Homological algebra – the study of complexes $\to V_i \xrightarrow{d} V_{i+1} \xrightarrow{d} V_{i+2} \to$ $d^2=0$

(V.S., modules, etc.)

Vital tool, most of us couldn't imagine life w/o it, since our first intro to it in Alg. Top.

But in earlier days it wasn't so obvious, people tried different things!

Alg Top: $X \rightsquigarrow C^*(X) \longrightarrow H^*(X)$ useful invts.

top space; $\mathbb{Z}$-linear complex using maps from simplices; $H^*(X) = \mathrm{Ker}\, d / \mathrm{Im}\, d$

differential is $d = \sum_i (-1)^i \mathrm{face}_i$

Annoying signs $\Rightarrow d^2=0$.

Mayer: Ignoring the signs you get $d^2 \neq 0$. But in char p, $d^p=0$. Called a p-complex. ^ $\mathrm{Ker}\, d^i / \mathrm{Im}\, d^{p-i}$ invts.

Spanier: Meh. These invariants can be recovered easily from usual $H^*(X)$.

So... people forgot about p-complexes. But it's time to bring 'em back $\rightsquigarrow$ "give invts of other stuff!!"

Both usual Hom. alg. and study of p-complexes are special cases of Hopfological alg. (Kh. 2005)

Outline: §1 Hom. alg. reminder, but recast in a new way! (Pareigis)

§2 Briefly, Hopf alg

§3 Key examples

§4 Statement of something cool.

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§1 Fix field $\Bbbk$, alg. $R/\Bbbk$. Usual setup: (Don't write in your notes) $R\text{-mod} \rightsquigarrow \mathrm{Kom}(R) \rightsquigarrow C(R) \rightsquigarrow D(R)$

② Abelian Cats (notion of s.e.s.) ⑤ Graded Ab Cats Ⓒ Triangulated Cats (notion of l.e.s.)

$$\begin{tikzcd}
\textcircled{0}\ R\text{-mod}_{(\text{f.g.})} \arrow[d, "\text{For}"'] \arrow[r, squiggly, "\textcircled{3}\ \text{allow bounded}", "\text{complexes}"'] & \textcircled{4}\ \mathrm{Kom}(R) \overset{\textcircled{4}}{=} A\text{-gmod} \arrow[d] \arrow[r, squiggly, "\textcircled{A}\ \text{take quotient by}", "\text{nullhomotopic maps}"'] & \textcircled{B}\ C(R) \arrow[d] \\
\textcircled{1}\ \Bbbk\text{-mod} & \mathrm{Kom}(\Bbbk) \overset{\textcircled{2}}{=} H\text{-gmod} & C(\Bbbk)
\end{tikzcd}$$

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⑥ Another way to view grading: ⑦ let $H = \Bbbk[\partial]/\partial^2$, graded w/ $\deg \partial = 1$.

⑧ let $A = R \otimes_{\Bbbk} H = R[\partial]/\partial^2$, $\deg R = 0$.

Reinterpret via H, A.

②

<u>Recall</u>: $f^\circ,$ is nulhomotopic if $\exists h$ s.t. $f = dh + hd$.

$$\begin{tikzcd} V_i \arrow[r] \arrow[d, "f"'] & V_{i+1} \arrow[r] \arrow[d, "f"] \arrow[dl, "h"'] & V_{i+2} \arrow[d, "f"] \arrow[dl, "h"'] \\ W_i \arrow[r] & W_{i+1} \arrow[r] & W_{i+2} \end{tikzcd}$$

How do we reinterpret using H, A ?

<u>Claim</u>: f is nulhomotopic iff it factors thru a projective H/A module.

<u>Ex</u>: A proj. H-mod is just a (gdd) free mod. $\bigoplus \; 0 \to \Bbbk \xrightarrow{\sim} \Bbbk \to 0$

When is a map

$$\begin{tikzcd} & 0 \arrow[r] & V \arrow[r] \arrow[dl, "h"'] \arrow[d, "f"] & 0 \\ & W \arrow[r] & W \arrow[r] & W \end{tikzcd}$$

nulhomotopic? $f = dh$ When

$$\begin{tikzcd} & 0 \arrow[r] & V \arrow[r] \arrow[d, "1"] & 0 & \\ 0 \arrow[r] & V \arrow[r, "\sim"] \arrow[d, "h"] & V \arrow[r] \arrow[d, "f"] & 0 & \in \mathrm{Proj}. \\ W \arrow[r] & W \arrow[r] & W & & \end{tikzcd}$$

<u>Def</u> For a $\overset{\text{gdd}}{\text{ring}}$ A, write <u>A-gmod</u> for its <u>stable category</u>, i.e. A-gmod / maps factoring thru projectives.

In <u>A-gmod</u>, $P \cong 0$ for any projective.

<u>Fact</u>: <u>A-gmod</u> is triangulated (in general). Tri. cats have shift functor, which for $K(R)$ looks like homological = grading shift. But that's just a coincidence!

For any <u>A-gmod</u>, define shift ΣV via $0 \to \Sigma V \to \underset{\text{proj}}{P} \to V \to 0$

well defined in <u>A-gmod</u> indep of choice of $P \to V \to 0$.

<u>Ex</u>: For C(R)

$$\begin{tikzcd} V = & 0 \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r] & 0 & \\ P \arrow[u, two heads] & & & V_2 \arrow[r, "\sim"] \arrow[u] & V_2 & \\ & & V_1 \arrow[r, "\sim"] \arrow[uu] & V_1 & & \\ \Sigma V \arrow[uuu] & & 0 \arrow[r] & V_1 \arrow[r] \arrow[u] & V_2 \arrow[r] \arrow[uu] & 0 \end{tikzcd}$$

agrees with grading shift.

Return to earlier page

$$\begin{tikzcd} ① \, C(R) = A\text{-}\underline{\text{gmod}} \arrow[rr, squiggly, "\overset{②}{\text{invert}}", "\text{qisoms}"'] \arrow[d] & & D(R) \, ③ \arrow[d] \\ C(\Bbbk) = H\text{-}\underline{\text{gmod}} \, ⑥ \arrow[rr, "\sim"] & & D(\Bbbk) \end{tikzcd}$$

<u>Recall</u>④ f is a qisom if $f^\bullet : H^i(V) \to H^i(W)$ is isom $\forall i$. ⑤ But in $C(\Bbbk)$, all qisoms are invertible, as all complexes are $\cong \bigoplus H^i(V)$

⑦ Equiv, f is a qisom if For(f). is an isom. (isom on underlying v.s.)

Can do this for any gdd alg $A/\Bbbk$, qisoms are localizing class in A-$\underline{\text{mod}}$?
Not sure, but so for the case we'll be restricting to soon.

<u>Final ingredient</u>: $\mathrm{Kom}(\Bbbk)$, $C(\Bbbk)$, $D(\Bbbk)$ is actually a $\otimes$-category.

Given V, W get a <u>bicomplex</u>

$$\begin{tikzcd}
{} & {} & {} & {} \\
V_0\otimes W_1 \arrow[u] \arrow[r] & V_1\otimes W_1 \arrow[u, "-1"'] \arrow[r] & {} \arrow[u] \arrow[r] & {} \arrow[u, "-1"'] \\
V_0\otimes W_0 \arrow[u] \arrow[r] & V_1\otimes W_0 \arrow[u, "-1"'] \arrow[r] & {} \arrow[u] & {} \arrow[u, "-1"']
\end{tikzcd}$$

sprinkle signs so that squares <u>anticommute</u>

↱ = − ⮥

It's <u>total complex</u> with $d = d_1 + d_2$ has $d^2 = 0$ b/c of signs.

Why does this work, in terms of H? B/c H is a <u>super-Hopf alg</u>.

i.e. $\exists$ alg map $H \xrightarrow{\Delta} H\otimes H$ saying how H acts on $V\otimes W$.
$d \mapsto d\otimes 1 + 1\otimes d$

alg map $\Rightarrow$ $0 = \Delta(d^2) = (d\otimes 1 + 1\otimes d)^2 = \underbrace{d^2\otimes 1}_{0} + \underbrace{1\otimes d^2}_{0} + \underbrace{(d\otimes 1)(1\otimes d)}_{= d\otimes d} + \underbrace{(1\otimes d)(d\otimes 1)}_{= -d\otimes d}$

why? $H\otimes H$ is alg via some <u>super</u> rule. Ignore the details ~~as~~ its new.

Similarly, $\mathrm{Kom}(\Bbbk) \underset{\otimes}{\circlearrowright} \mathrm{Kom}(R)$ in same way. This is b/c A is a <u>Hopf-comod-alg</u> over H.

Return to big chart. (Bottom $\otimes$) $\circlearrowright$ Top.

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§2 Let H be any fin.dim (normal or super) Hopf alg $/\Bbbk$.
$A = R \otimes_{\Bbbk} H$ is a H-comod-alg.

$$R\text{-mod} \xrightarrow[\text{“}H\text{-complexes”}]{\text{allow}} A\text{-gmod} \rightsquigarrow \underline{A\text{-gmod}} \rightsquigarrow D(A)$$

<u>Hopfological Algebra</u>

Kh '05, Kh+Qi, Qi

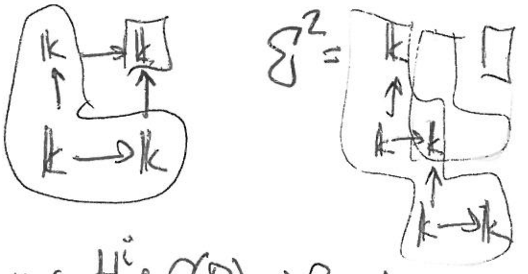
$\boxed{\text{Optional}}$ Ex: $H = \Bbbk[\partial_1, \partial_2] / \partial_1\partial_2 = -\partial_2\partial_1$, $\partial_1^2 = \partial_2^2 = 0$ $\boxed{\text{super Hopf}}$

H-mod = bicomplexes w/ diagonal grading

Now Σ is NOT grading shift! $\Sigma\left(\begin{tikzcd}[row sep=small, column sep=small] & 0 & \\ 0 \arrow[r] & \Bbbk \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & \end{tikzcd}\right) = ?$

$$\text{Proj} = \bigoplus \begin{tikzcd} \Bbbk \arrow[r] & \Bbbk \\ \Bbbk \arrow[r, "1"'] \arrow[u, "1"] & \Bbbk \arrow[u, "-1"'] \end{tikzcd}$$

Use colored chalk



Rmk: One big difference b/w Hom alg + Hopf alg:

Hom alg has built in notion of a <u>t-structure</u>, gives you $H^i : C(R) \to R\text{-mod}$

But A-gmod does not. Several functors $C(H) \to \Bbbk\text{-mod}$

- horiz cohom
- vert cohom
- total cohom

but not t-structures in usual sense.

Comparing these allows one to study spectral sequences!!!

Key Ex: $H = \Bbbk[\partial]/\partial^p$ non-super $\Delta(d) = d \otimes 1 + 1 \otimes d$ so $0 = \Delta(d^p) = d^p \otimes 1 + 1 \otimes d^p$ + other terms

Uh oh... not Hopf... but if char $\Bbbk = p$ then other terms vanish!!! ☑

Def: A p-complex is an object of H-gmod, i.e. $V \xrightarrow{d} V \xrightarrow{d} \cdots$ w/ $d^p = 0$.

Projectives $\bigoplus \underbrace{\Bbbk \to \Bbbk \to \Bbbk \to \cdots \to \Bbbk}_{p}$

Σ is not grading shift. $\Sigma\, 0 \to \Bbbk \to 0$ is (Use colored chalk) $\underbrace{\Bbbk \to \cdots \Bbbk}_{p-1}$.

$\boxed{\text{If optional example: many functors } C(H) \to \Bbbk\text{-mod} \quad \operatorname{Ker} \partial^i / \operatorname{Im} \partial^{p-i} \ldots}$

§2 redux One more bit of hom. alg. A <u>dg-alg</u> A is a graded alg with map $A \xrightarrow{d} A$ of degree 1 satisfying ① $d(fg) = d(f)g + (-1)^{\deg f} f d(g)$ (super)

② $d^2 = 0$

$$A\text{-grmod} \rightsquigarrow C(A) \rightsquigarrow D(A)$$

$\boxed{\text{before, } A = R[\partial]/\partial^2. \quad d = \text{mult by } \partial}$

Can generalize this too, to any H-comod-alg. When $H = \Bbbk[\partial]/\partial^p$, char $\Bbbk = p$ get a <u>p-dg-alg</u>: graded alg A w/ $A \xrightarrow{d} A$ s.t. ① $d(fg) = d(f)g + f d(g)$

② $d^p = 0$.

<u>Ex:</u> $A = \Bbbk[x_1, x_2, \ldots, x_n]$ $d(x_i) = x_i^2$. $\Longrightarrow d(x_i^n) = n x_i^{n+1}$

$$\Longrightarrow d^p(x_i^n) = n(n+1)(n+2)\ldots(n+p-1)\, x_i^{n+p} \equiv 0 \text{ mod } p.$$

One differential to rule all characteristics, a common feature.

<u>Exercise!</u> $A \cong \Bbbk$ as stably p-complexes $\langle x_1, x_1^2, \ldots x_1^p \rangle$ is <u>contractible</u> (projective)

$\Longrightarrow \boxed{D(A) \cong D(\Bbbk)}$

<u>Exercise!</u> Compute $d(e_i)$. Show A^{S_n} also p-dg-alg, also $\cong \Bbbk$

<u>§4 Grothendieck group.</u>

For ab. cat $\mathcal{A}$, $[\mathcal{A}] \overset{\wedge}{=} \mathbb{Z}\langle [M] \rangle \Big/ \begin{matrix} [M]+[N]=[L] \text{ in ses} \\ 0 \to M \to L \to N \to 0 \end{matrix}$

$[\Bbbk\text{-mod}] \cong \mathbb{Z}$, $\mathbb{Z}$ action on $[R\text{-mod}]$ is induced from $\circlearrowleft \otimes$.

For <u>graded</u> ab cat, has $\mathbb{Z}[q, q^{-1}]$-mod. structure via $q[V] = [V(1)]$. (Take sum over complex.)

For tri cat, $[\mathcal{A}] \overset{\wedge}{=} \mathbb{Z}\langle [M] \rangle \Big/ \text{l.e.s.} \Longrightarrow [\Sigma M] = -[M]$. [redacted]

So in usual hom alg, $\Sigma = (1)$ and thus $q = -1$. Follows from $P \simeq 0$ $[V \xrightarrow{\sim} V] = 0$ $\overset{\shortparallel}{[V] + q[V]}$

$\boxed{[C(\Bbbk)] = \mathbb{Z}[q, q^{-1}]/q=-1}$

But for p-complexes, $\Sigma \neq (1)$. Instead, $[\underset{n}{\underline{\Bbbk}} \to \cdots \to \Bbbk] = 0$

$$(1 + q + q^2 + \ldots + q^{p-1})[\Bbbk] \Longrightarrow q = \zeta_p \text{ !!!}$$

So for any p-dg-alg, $\begin{matrix}[\mathcal{C}(A)]\\ [\mathcal{D}(A)]\end{matrix}$ is a module over $[\mathcal{C}(H)] \cong \mathbb{Z}[\zeta_p]$.

Given usual graded alg A catifying your favorite $\mathbb{Z}[q,q^{-1}]$-module M (i.e. $[\mathcal{D}(A)] \cong M$), try to equip it with differential so that in char p one has $[\mathcal{D}(A)] \cong M \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[\zeta_p]$.

Thm: ☑ for catfns of quantum $\mathfrak{sl}_2$ + repns, also $U_q^+(\mathfrak{g})$.
(Q-K, E-Q) soon for —"— Hecke algs