Today's goal: Introduce the main players - Soergel bimodules, a cat^n of $\mathbb{H}_W$.
Easy to get a handle on since alg and combinatorial.

§1 | Reflection Reps + Polys | Fix (W,S). Define the symmetric Cartan Matrix of (W,S) to be
$$A = \begin{pmatrix} 2 & & a_{st} \\ & 2 & \\ a_{t,s} & & \ddots \\ & & & 2 \end{pmatrix}$$
with $a_{s,t} = a_{t,s} = -2\cos\frac{\pi}{m_{st}}$ (when $m_{st} < \infty$)
(when $m_{st} = \infty$ can use $a_{s,t} = a_{t,s} = \pm 2$ or anything suitably generic)

Let $\mathfrak{h}^*/\mathbb{R}$ have basis $\{\alpha_s\}_{s\in S}$, called simple roots.

$W \circlearrowright \mathfrak{h}^*$ by $s(\alpha_t) = \alpha_t - a_{st}\alpha_s$, so $s(\alpha_s) = -\alpha_s$, $t \neq s$: $s(\alpha_t) = \alpha_t + 2\cos\frac{\pi}{m_{st}}\alpha_s$

Rmk: Many ways to generalize, see exercises.

Def: Let $R = \mathrm{Sym}^*(\mathfrak{h}^*) = \mathbb{R}[\alpha_s] \circlearrowleft W$. Graded, $\deg \alpha_s = 2$.
For $I \subset S$, let $R^I = R^{W_I}$ invts under parabolic subgp W_I.

Ex: $W = S_n \circlearrowright \mathbb{R}[x_1,\dots,x_n]/\sum x_i = 0$ ← can ignore. $\alpha_i = x_i - x_{i+1}$

$W_I = S_3 \times S_1 \times S_2 \times \dots$ $R^I = \mathbb{R}\left[\begin{matrix} x_1+x_2+x_3 \\ x_1x_2+x_1x_3+x_2x_3 \\ x_1x_2x_3 \end{matrix}, x_4, \begin{matrix} x_5+x_6 \\ x_5x_6 \end{matrix}, \dots\right] / \sum x_i = 0$

Thm (Chevalley): Suppose W_I is finite $\leftrightarrow$ I is finitary. Then
R^I is a poly ring of some transcendence degree $|S|$), generated by alg ind polys in various degrees determined by W_I. (Many facts, see Humphreys "Cox. Gps..." $\prod d_i = |W_I|$, $\sum d_i = \dots$)

So the rings R^I aren't so bad. (Compare to other invariant subrings!)
But what is even better is the relationship of R^I to R. Think of this as a souped up Chev Thm.

Thm: $R^I \subset R$ is a Frobenius extension. So is $R^J \subset R^I$ for $I \subset J$.

Def: A (commutative) ring ext. $A \subset B$ is a Frob Ext if it is equipped w/ $\partial: B \to A$, A-linear, and if B is free over A w/ dual bases $\{b_i\}$ and $\{b_i^*\}$ s.t. $\partial(b_i b_j^*) = \delta_{ij}$.
When A, B are graded rings, require dual bases to be homogeneous, and $\deg \partial = -2\ell$ then called Frob Ext of degree ℓ.

Why these, and why "Frobenius" – comes from Frob Reciprocity. **Ex:** $H \subset G$ $\quad \mathbb{C}[H] \subset \mathbb{C}[G]$ ②

The bimod. ${}_B B_A$ gives functor ${}_B B_A \otimes_A \bullet$: A-mod $\to$ B-mod <u>Induction</u>

${}_A B_B$ —— // —— B-mod $\to$ A-mod <u>Restriction</u>

For <u>any</u> ring ext $\quad \mathrm{Ind} \dashv \mathrm{Res}$ ie $\mathrm{Hom}_B(\mathrm{Ind}\,M, N) \cong \mathrm{Hom}_A(M, \mathrm{Res}\,N)$

dets by unit + counit of adjunction $\quad \mathrm{Hom}_B(\mathrm{Ind}\,\mathrm{Res}\,M, M) \cong \mathrm{Hom}_A(\mathrm{Res}\,M, \mathrm{Res}\,M) \ni 1_M$

get natl trans $\quad \mathrm{Ind}\,\mathrm{Res} \longrightarrow \mathbb{1}_{B\text{-mod}}$

Counit $\quad {}_B B \otimes_A B_B \longrightarrow B \quad$ just multiplication

Sim, unit is $\quad \mathbb{1}_{A\text{-mod}} \longrightarrow \mathrm{Res} \circ \mathrm{Ind}$

${}_A A_A \longrightarrow {}_A B_A \quad$ just inclusion

satisfy some natural conditions, spelled out later

For Frob ext, $\mathrm{Res} \dashv \mathrm{Ind}$. Now have maps in other direction

${}_A B_A \longrightarrow {}_A A_A \quad \partial$

${}_B B_B \longrightarrow {}_B B \otimes_A B_B \quad \Delta(1) = \sum b_i \otimes b_i^* \quad$ indep of choice of dual bases

With the grading included, $\quad \mathrm{Ind} \dashv \mathrm{Res} \dashv \mathrm{Ind}(2l)$, but have nice intermediate description –

consider functors Ind and $\mathrm{Res}(l)$

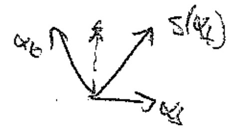
$B \otimes_A B(l) \longrightarrow B \quad$ degree $+l$

$B \longrightarrow B \otimes_A B(l) \quad$ degree $+l$

$B(l) \longrightarrow A \quad$ degree $-l$

$A \longrightarrow B(l) \quad$ degree $-l$

<u>Let's do it in examples:</u> $\quad R^S \subset R$. Now $\alpha_s^2 \in R^S$.



$\alpha_t \leftrightarrow \alpha_t - 2\cos\frac{\pi}{m}\alpha_s$

so $\alpha_t + \cos\frac{\pi}{m}\alpha_s \in R^S$

<u>Claim:</u> $R^S = \mathbb{R}[\alpha_s^2, \{\alpha_t + \cos\frac{\pi}{m_{st}}\alpha_s\}_{t \neq s}]$

<u>Def:</u> $\partial_s : R \to R^S \quad$ <u>Demazure</u> or <u>divided difference operator</u> $\quad \deg = -2$

$\partial_s(f) = \dfrac{f - s(f)}{\alpha_s}$. Clearly in $\mathrm{f.f.}(R)^S$. Numerator in $R^{-S} = \{f \in R \mid s(f) = -f\}$

<u>Claim:</u> $R^{-S} = R^S \cdot \alpha_s$.

Check: ∂_s is R^S-linear $\quad$ <u>Fact:</u> $\partial_s(R^S) = 0$. $\quad$ Check: $\{1, \frac{\alpha_s}{2}\}$ and $\{\frac{\alpha_s}{2}, 1\}$ are dual bases

~~Check~~ $\quad$ <u>Note:</u> $\partial_s(\alpha_t) = a_{st}$. $\leftarrow$ "Cartan matrix encodes Frob. ext. structures"

More about ∂_s: ① Twisted Leibniz Rule: $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$

② Braid relation $\partial_s\partial_t \cdots = \partial_t\partial_s \cdots$ so $\partial_{\underline{w}} = \partial_w$ for rex, indep of choice of rex.

③ $R \cong R^s \oplus R^{-s} \underset{R^s\text{-bim}}{\cong} R^s \oplus R^s(-2)$

$R^{-s} = R^s\alpha_s$

$f = (g,h)$ i.e. $f = g + h\frac{\alpha_s}{2}$ for $g, h \in R^s$ $\quad h = \partial_s(f)$, $\quad g = \partial_s\left(\frac{f\alpha_s}{2}\right)$

Ex ∂: W Weyl gp. $R/R_+^W \cong H^*(Fl) \cong C$ finite dim ring. Has trace map

$\partial_{w_0} = \partial : C \longrightarrow \mathbb{R}$ integrate against top class (general for coh. of sm. proj. vty)

Dual bases given by Schubert calculus. Top class is $\prod_{\alpha \in \Phi_+ \text{ pos roots}} \alpha$, one basis given by $\partial_w\left(\prod \alpha\right)$.

We're interested in relative version, $\underset{H_G^*(pt)}{R^W} \subset \underset{H_B^*(pt)}{R}$, have $\partial_W = \partial_{w_0} : R \longrightarrow R^W$

however, Schubert basis no longer gives dual bases! $\partial_{w_0}(\sigma_i \sigma_j^*) \in R_+^W$ but not nec. 0.

No nice closed formulas known!!

Open Problem: Find Root-theoretic descriptions of dual bases for $R^I \subset R^J$, for all Cox gps

§3 S. Bim

Def: $B_s = R \otimes_{R^s} R(1)$ as R-bimod $\quad$ Ind ∘ Res(1) self-biadjoint.

means $1 \otimes 1$ lies in degree -1.

Visualize as sum of $f_1 \,|\, f_2$. R^s can cross through, means $\otimes 1$

If $f = g + h\frac{\alpha_s}{2}$ for $g, h \in R^s$ then $f\,| = |\, g + \frac{\alpha_s}{2}\,|\, h$

So as right R-mod, B_s has basis $\left\{ \underset{1\otimes 1}{|} \text{ and } \underset{\frac{\alpha_s}{2}\otimes 1}{\frac{\alpha_s}{2}|} \right\}$

Exercise: Why free?

Def: A Bott-Samelson bimod is $BS(\underline{w}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u = R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R(d)$

$f_1 \,|\, f_2 \cdots \,|\, f_{d+1}$

Exercise: $BS(\underline{w})$ has basis $\left\{ \alpha_{s_1}^{\varepsilon_1} \otimes \alpha_{s_2}^{\varepsilon_2} \otimes \cdots \otimes \alpha_{s_d}^{\varepsilon_d} \otimes 1 \right\}_{\underline{\varepsilon} \in \{0,1\}^d}$ as right R-mod

Examination: 1 $B_s\otimes B_s = R\otimes_{R^s} R\otimes_{R^s} R(2) \cong R\otimes_{R^s}(R^s\oplus R^s(-2))\otimes_{R^s} R(2) = R\otimes_{R^s}R(0)\oplus R\otimes_{R^s}R(2)$

$= B_s(-1)\oplus B_s(+1)$

$f\otimes g\otimes h \longmapsto (f\,\partial_s(g)\otimes h,\ f\,\partial_s(g\tfrac{\alpha_s}{2})\otimes h)$

$(a\otimes\tfrac{\alpha_s}{2}\otimes b + c\otimes 1\otimes d) \longleftarrow (a\otimes b,\ c\otimes d)$

Categorifies $\underline{H}_s\underline{H}_s = \underline{H}_s v^{-1} + \underline{H}_s v$

2 $B_s\otimes B_t$ can slide f_2 out of middle since R^s and R^t generate R. (unless $a_{st}=2$)

$f_1 | f_2 \} f_3$ So as bimod, generated by $1\otimes 1\otimes 1$

$\exists$ surjective map $R\otimes_{R^{st}} R(2) \longrightarrow B_s\otimes B_t$, $f\otimes g \longmapsto f\otimes 1\otimes g$.

Is there an inverse map?? When $m_{st}=2$ yes. $\partial_s\partial_t=\partial_t\partial_s$. Else no.

What to do with $1\otimes f\otimes 1$? Unique way to slide out??

$m_{st}=2$ catfys $\underline{H}_s\underline{H}_t=\underline{H}_t\underline{H}_s$

3 $\underbrace{B_s\otimes B_t\otimes\cdots}_{m_{st}}$ has $R\otimes_{R^{st}}R(m_{st})$ as a summand!

4 $R\xrightarrow{\Delta} B_s$ unique linear elt (up to scalar) w/ $f c_s = c_s f$.

$1\longmapsto \frac{\alpha_s}{2}\otimes 1 + 1\otimes\frac{\alpha_s}{2} = c_s$

draw as $\boxed{c_s}$ since creates a gap for f to slide thru.

5 $B_s\otimes B_t\otimes B_s$ $m_{st}=3$.

slide all s-t-inv out; slide all out

gen as R-bim by $\twoheadleftarrow R\otimes_{R^{sts}}R(3)$

and $\boxed{c_t} \cong B_s$

Claim: $B_s\otimes B_t\otimes B_s \cong B_s\oplus R\otimes_{R^{st}}R(3)$

Catfys $\underline{H}_s\underline{H}_t\underline{H}_s = \underline{H}_s + \underline{H}_{sts}$

Thm (Soergel): $\mathbb{H}_W \longrightarrow [R\text{-bimod}]$ is an injective homomorphism of $\mathbb{Z}[v^{\pm1}]$-alg

$\underline{H}_s \longmapsto [B_s]$

More categorical version:

Def: A <u>Soergel bim</u> is a $(\oplus, \otimes, (n))$ of a <u>summand</u> of a BS Bim. Form a full add monoidal subcat of R-bim

Thm (SCT): ① $\exists!$ indec. summand $B_{\underline{w}} \overset{\oplus}{\subset} BS(\underline{w})$ which does not appear in $BS(\underline{y})$ for shorter $\underline{y}$

② $\exists$ canonical isom $B_{\underline{w}} \cong B_{\underline{w}'}$ when $\underline{w} = \underline{w}'$. So just write as B_w.

③ $\{B_w(n)\}_{n \in \mathbb{Z}}$ forms complete list of non isom indec. in SBim.

$\Rightarrow [SBim] = \mathbb{Z}[v^{\pm}]\langle [B_w] \rangle$ ← basis

④ $H_W \xrightarrow{\sim} [SBim]$ is an isom. (upper triangularity)

⑤** $\mathrm{Hom}(B, B')$ is free as right R-mod w/ graded rank $([B], [B'])$. Soergel Hom Formula.

Ex:

- $\mathrm{Hom}(R, R) = R$ $\quad (1, 1) = 1$
- $\mathrm{Hom}(R, B_s) = R(-1)$ $\quad (1, \underline{H}_s) = \varepsilon(H_s + v) = v$
- $\mathrm{Hom}(B_s, B_s) = R \oplus R(-2)$ $\quad (\underline{H}_s, \underline{H}_s) = (1, \underline{H}_s^2) = \varepsilon(\underline{H}_s^2) = \varepsilon((v+v^{-1})\underline{H}_s) = v^2 + 1$.

  identity and left mult by α_s

- $\mathrm{Hom}(B_s B_s, B_s)$ has gd rk $\boxed{v^{-1}} + 2v + v^3$

  ← maps of negative degree.

  proj to $B_s(1)$ and $B_s(-1)$

  then left mult by α_s