Recall our grand induction: $B_x \overset{\oplus}{\subset} B\cancel{S}(\underline{x})$ ($\overline{B_x}$), has induced form has $L = \rho\circ$ / $B_xB_s \overset{\oplus}{\subset} B\cancel{S}B_s$, has induced form has $L_\varsigma = \rho\circ + \varsigma\, id_{B_x}\rho\circ$

Suppose "everything" for $< xs$. WTS

$$hL(x;s)_\varsigma \ \ \varsigma \geq 0 \xrightarrow[\text{limiting argument}]{} HR(x;s)_\varsigma \ \ \varsigma \geq 0 \xRightarrow{\varsigma = 0} S(xs),\ hL(xs),\ HR(xs)$$

$HR(x) \Rightarrow HR(x;s)_{\varsigma >> 0}$

How to finish the loop?

(2) $S(\leq xs) \Longrightarrow F_x$ has diagonal miracle, so $F_x^j = \bigoplus_{z \in ()} B_z(j) \overset{\oplus}{\subset} F_{\underline{x}}^j = \bigoplus_{\underline{y} = \underline{x}_j} BS(\underline{y})(j)$

⇓

(1) $hL(<xs)$, $HR(<xs) \Longrightarrow$ (3) ~~[illegible]~~ $RoHR(x) := F_x^j(-j)$ satisfies HR, hL using induced form, L.

⇓

$hL(<x;s)$, $HR(<x;s) \Longrightarrow$ (4) We know a LOT about $F_xF_s = B_xB_s \to \boxed{\ } \to$

enough to deduce $hL(x;s)_\varsigma$

Need to explain 3 things:
(1) How we get $RoHR$ (it's a similar argument)
(2) Why F_xF_s has anything to do w/ $hL(x;s)_\varsigma$
(3) Using facts about F_xF_s to deduce $hL(x;s)_\varsigma$

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(1) **Prop 1:** $S(\leq x)$, $HR(y;s)$ $y < x$ $\Longrightarrow RoHR(x)$

Pf: $F_x \overset{\oplus}{\subset} F_yF_s$ so $F_x^j \overset{\oplus}{\subset} F_y^jB_s \oplus F_y^{j-1}(1)$ (L-stable)

$y = xs < x$. $\Rightarrow F_x^j(-j) \overset{\oplus}{\subset} F_y^jB_s(-j) \oplus F_y^{j-1}(-(j-1))$ ← Has HR

← has HR?? Not really, but...

$$F_y(-j) = \bigoplus B_z = B^\uparrow \oplus B^\downarrow \quad \text{where} \quad B^\uparrow = \bigoplus_{zs > z} B_z \quad B^\downarrow = \bigoplus_{zs < z} B_z$$

Now $\overline{B^\uparrow B_s}$ has HR by $HR(y,s)$. OTOH $B^\downarrow B_s \cong B^\downarrow(1) \oplus B^\downarrow(-1)$ L-stable decomp.

Should think that form on $B^\downarrow B_s$ is nondeg but pairs $B^\downarrow(1)$ against $B^\downarrow(-1)$. Regardless, it's clear that $\langle , \rangle_{B^\downarrow B_s}\big|_{B^\downarrow(1)} = 0$ for degree reasons (V has $hL \Longrightarrow V(k)$ has $(,)_L = 0$, $k > 0$.)

The map $F_x^j(-j) \longrightarrow B^\downarrow(1) \oplus B^\downarrow(-1)$ lands entirely inside $B^\downarrow(1)$ (NO maps $B_z \to B_z(-1)$ we have S. Conj) and won't contribute to the Lefschetz pairing!

$\overline{F_x^j(-j)} \xrightarrow[\text{project}]{} \overline{B^\dagger B_s} \oplus \overline{F_y^{j-1}(-(j-1))}$ is an isometry of Lef forms (2)

also ~~...~~ injective – the map $F_x^j(-j) \to B^\downarrow(1)$ is in the "maxl ideal", so the rest of the projection must be ~~...~~ injective (split)

So $\overline{F_x^j(-j)}$ is L-stable summand of HR w/ balanced ~~...~~ $\overset{\text{gdd dim}}{}$ $\Rightarrow$ HR.

Key idea: $F_y B_s$ didn't have HR, wasn't perverse (at least, until you look at homotopy) but Lefschetz form don't care! Image had HR. Similar tricks to come.

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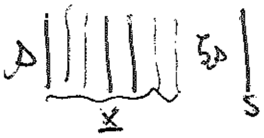
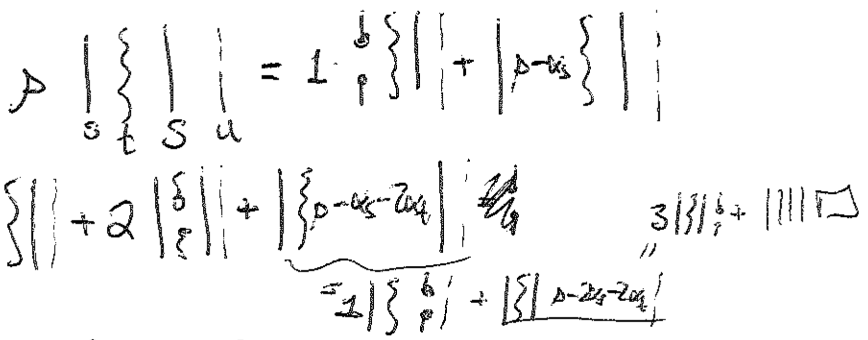
(2) Consider L_ζ on $BS(\underline{x})B_s$  $xs > x$

Multiplication by this is equal to the endomorphism $\sum_i \lambda_i$ 

for some scalars λ_i and linear poly f.

Ex: $\rho = \omega_s + \omega_t + \omega_u$



If had ζ, would get $3+\zeta$ for last coeff.

Claim: $\lambda_i > 0$ This was on the exercises. If $wt > w$ ~~...~~ then $\langle w\rho, \alpha_t^\vee \rangle > 0$

also $\langle \zeta\rho, \alpha_s^\vee \rangle > 0$ for $\zeta > 0$.

Recall geordie's Weak Lefschetz Substitute – ~~...~~ to show hL on V we want a map of degree +1 $V \xrightarrow{\Phi} W$, ~~...~~ commuting w/ L, s.t. $\langle v, Lw \rangle = \langle \Phi v, \Phi w \rangle$ and Φ injective in degrees < 0.

(W has HR)

Well, we have $BS(\underline{x})B_s \xrightarrow{\sum_i \text{||||pill}} \bigoplus_{\underline{y} = \underline{xs}\setminus 1} BS(\underline{y})$. Not quite right – stupid scalars muck it up.

$\sum \langle \text{||||pill}(v), \text{||||pill}(w) \rangle = \sum \langle v, \text{||||}\,\overset{b}{p}\,\text{||||}w \rangle$, want $\sum \lambda_i \langle v, \text{||||}\,\overset{b}{p}\,\text{||||}w \rangle$.

2 choices. (1) Let $\Phi = \sum \sqrt{\lambda_i}\ \text{||||pill}$ works since $\lambda_i > 0$.

(2) Let $\Phi = \sum$ ||||pill coeff 1, but renormalize $\langle , \rangle$ on $BS(\underline{y})$ by $\lambda_i > 0$ ← won't screw up HR

Why is Φ injective in degrees <0? B/c Φ is first differential in F_xF_s! (or renormalized version) ③

$BS(\underline{x})B_s \xrightarrow{\sum \|\|\|\|} \bigoplus_{\underline{y} \subset \underline{x}s} BS(\underline{y})$ ← no shift, $F_{xs}^1(-1)$, and cohomology was $R_{xs}(-\ell(xs))$, injective in degrees $<\ell(xs)$ even

But why would $BS(\underline{y})$ have HR. It doesn't... Can do something like in Prop 1.

$\underline{\text{Thm}}$: $R_0HR(x) \Rightarrow hL(x,s)_\zeta$ for $\zeta \geq 0$, $xs > x$, and also $hL(x,s)_\zeta$ for $\zeta > 0$, $xs < x$.
etc.

$\underline{\text{Pf}}$: ① $\zeta > 0$, $xs < x$. Don't need Ro. Comp at all, fix a basis + compute, exercise. Gives hL. HR is a limit exercise

② $\zeta > 0$, $xs > x$. We have diagonal miracle for F_x. $F_x^0 = B_x$

$F_x^1 = \bigoplus_{z<x} B_z(1) = B^\uparrow(1) \oplus B^\downarrow(1)$

$$\overline{(F_xF_s)^0} \xrightarrow{\overline{\Phi}} \overline{(F_xF_s)^1} = \overline{B^\uparrow B_s} \oplus \overline{B^\downarrow B_s} \oplus \overline{B_x} \quad \boxed{\checkmark}$$

$\overline{(F_xF_s)^0} = \overline{B_xB_s}$

has HR by $HR(z,s)_\zeta$, $z<x$ — has HR by part ① — has HR wrt L (no ζ term)

$\underline{\text{Rmk}}$: Can't split $B^\downarrow B_s$ into $B^\downarrow(1) \oplus B^\downarrow(-1)$ as before. Splitting does not commute w/ L_ζ. (What is L_ζ on the RHS?) only have L, want J_ρ not to contribute

→ maps are

③ $\zeta = 0$ $xs > x$ $\quad \overline{B_xB_s} \xrightarrow{\overline{\Phi}} \overline{B^\uparrow B_s} \oplus \overline{B^\downarrow B_s} \oplus \overline{B_x}$

$L_\zeta = L$. $\quad \overline{B^\downarrow B_s} \cong \overline{B^\downarrow(1)} \oplus \overline{B^\downarrow(-1)}$ ← now L commutes w/ decomp!

Can't quite use the same trick immediately to finish. ~~[illegible]~~ Before it was a degree 0 map. Now degree 1, can hit $B^\downarrow(-1)$. And will – that's the shift. That's split off in $B_xB_s = B_{xs} \oplus$ rest!!

~~[illegible]~~

~~[illegible]~~

Now $F_xF_s \in K^{\geq 0} \Rightarrow$ only neg shift allowed in minimal complex $\Rightarrow$ the $\overline{B^\downarrow}(1)$ term must contract against something in degree 2. We can ignore it.

$$\overline{B_xB_s} \xrightarrow{\overline{\Phi}} \overline{B^\downarrow(-1)} \oplus \left(\overline{B^\uparrow B_s} \oplus \overline{B_x}\right)$$

← has HR for L w/ shift. ← has HR for L ~~[illegible]~~

ⓐ If $\overline{B_xB_s} \to \overline{B^\downarrow(-1)}$ nonzero, hL holds. $L^k\overline{\Phi}(v) \neq 0$ by hL on $B^\downarrow$, so $L^kv \neq 0$.
deg $v = -k$, The stupid shift!

ⓑ If $\overline{B_xB_s} \to \overline{B^\downarrow(-1)}$ zero, then $\overline{\Phi}$ goes into something w/ HR. $\boxed{\checkmark}$