

Outline of proof of Serre conjecture/recap:

$S(w) := [B_w] = H_w$. If true, then $\exists!$ non-zero invariant \langle, \rangle on B_w , up to scalar, and it is nondegenerate.

$\langle, \rangle_{BS(w)}|_{B_w}$ is non-zero, so nondeg.

$hL(w) := \overline{B_w}$ with $\langle, \rangle_{BS(w)}|_{B_w}$ and left action given by left mult by ρ with $\partial_s(\rho) > 0 \forall s$

$hL(w) := hL(w) \uparrow \text{res}$ satisfies hL.

$HR(w) := \text{---} HR$. Remark: $\langle, \rangle_{BS(w)}$ already normalized st. $\langle c_{\text{bot}}, c_{\text{bot}} \rangle_{B_w}^{-l(w)} > 0$.

As discussed, for induction we want to investigate the semismall object $\overline{B_w B_s}$, when $w \leq s$.

$hL(w, s) := \overline{B_w B_s}$ w/ $\langle, \rangle_{BS(ws)}|_{B_w B_s}$...

$HR(w, s) := \text{---}$

There will be yet more, but enough for now.

Assume $S(y) \ hL(y) \ HR(y) \ \forall y < ws$ (including $y=w$)

Then $H_w H_s = H_{ws} + \sum \mu(w, s, y) H_y$ and $\mu(w, s, y) = \dim \text{Hom}^0(B_w B_s, B_y) = \dim \text{Hom}^0(B_y, B_w B_s)$ and no negative degree maps.

The pairing $\text{Hom}^0(B_y, B_w B_s) \times \text{Hom}^0(B_w B_s, B_y) \rightarrow \text{End}^0(B_y) = \mathbb{R}$ is the LI Pairing

Both B_y and $B_w B_s$ have non-deg forms $\langle, \rangle_{B_y} \ \langle, \rangle_{B_w B_s}$ ← see exercises

so given $\psi \in \text{Hom}(B_y, B_w B_s)$, its adjoint $\psi^*: B_w B_s \rightarrow B_y$ satisfies $\langle \psi(b), b' \rangle_{B_w B_s} = \langle b, \psi^*(b') \rangle_{B_y}$

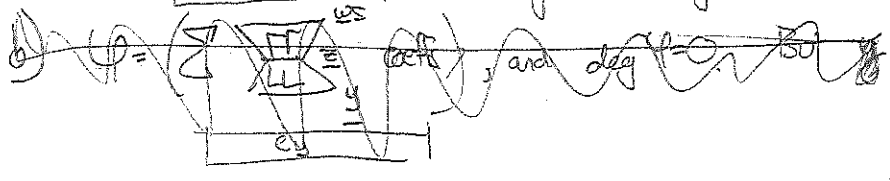
This gives identification $\text{Hom}^0(B_w B_s, B_y) \cong \text{Hom}^0(B_y, B_w B_s)$ allowing transfer of LIP to LIF

$(\psi, \psi)_y^{w, s} = \text{coeff of } 1 \text{ in } \psi^* \psi$.

Embedding Thm: $\text{Hom}(B_y, B_w B_s) \rightarrow \overline{B_w B_s}$ has image inside $P^{-l(y)}$, is injective, $\psi \mapsto \overline{\psi(c_{\text{bot}})}$ is isometry up to pos. scalar.

Pf: a) Clearly $\Delta^{l(y)+1} c_{\text{bot}} = 0$ in B_y , for degree reasons. So $\Delta^{l(y)+1} \overline{\psi(c_{\text{bot}})} = 0$.


Lies in degree $\deg c_{\text{bot}} = -l(y)$.



b) Any map $BS(y) \rightarrow BS(ws)$ is $\sum \frac{ws}{y} \frac{\prod}{L_1} \stackrel{\text{coeff}}{=} \dots$ ignoring choice of re . (2)
 Actually, thinking of factory thru B_z .

But we're interested in $B_y \rightarrow B_w B_s$.
 For B_y , B_y semismall, \exists no deg < 0 maps to B_z
 \exists deg 0 iff $y=z$.

B_y semismall $B_w B_s$, \exists no deg < 0 maps from B_z .

\sum  unless $y=z, L_1 = 1, \text{deg } \Pi^2 = 0, \text{coeff} = \text{scalar}$.

We've seen that $\frac{ws}{z} \frac{\prod}{L_1} (c_{\text{bot}})$ form a basis, so $\overline{\varphi}(c_{\text{bot}}) = 0 \Rightarrow c_{\text{bot}} = 0$.

c) We know $\langle c_{\text{bot}}, c_{\text{top}} \rangle_{B_y} = 1$ and $\langle c_{\text{bot}}, \mathcal{P}^{(y)} c_{\text{bot}} \rangle = N > 0$. Thus

$$(\varphi, \psi)_y^{ws} = \text{coeff of } 1 \text{ in } \psi \circ \varphi = \langle \psi \circ \varphi(c_{\text{bot}}), c_{\text{top}} \rangle = \frac{1}{N} \langle \psi \circ \varphi(c_{\text{bot}}), \mathcal{P}^{(y)} c_{\text{bot}} \rangle$$

$$\frac{1}{N} (\overline{\varphi}(c_{\text{bot}}), \overline{\psi}(c_{\text{bot}}))_{\mathcal{P}}^{-l(y)} = \frac{1}{N} \langle \varphi(c_{\text{bot}}), \psi(\mathcal{P}^{(y)} c_{\text{bot}}) \rangle$$

Cor: $HR(w, s) \Rightarrow LIF$ is non-deg $\Rightarrow B_y$ has correct multiplicity in $B_w B_s$ $\forall y \Rightarrow [B_{ws}] = H_{ws}$

Also, $B_{ws} \circ B_w B_s$ is preserved by \mathcal{P} , restriction of HR has HR (so long as restriction of \langle, \rangle is nondeg) $\Rightarrow hL(ws), HR(ws)$.
 which it is by $S(ws)$.

\exists ITS $S(\leftarrow ws), hL(\leftarrow ws), HR(\leftarrow ws) \Rightarrow HR(w, s)$.

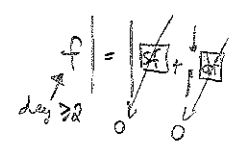
We use a deformation argument. Let $L_z \subset B_w B_s$ be given by $\mathcal{P}[\overline{w}] + [\overline{s}]$.
 $\mathbb{S} = \mathbb{R}$.

Then $hL(w, s)_z :=$ same but for L_z , not $\mathcal{P} = L_0$.
 $HR(w, s)_z$

Can even define these for case when $ws < w!$ Exercise: $hL(w) \Rightarrow hL(w, s)_z$ when $ws < w$ and $\mathbb{S} \neq 0$.

Prop: $HR(w) \Rightarrow HR(w, s)_z$ for $\mathbb{S} \gg 0$. PF: ~~...~~

$L_z = \mathcal{P} + \mathbb{S}L$, so L_z^k has binomial expansion. However, $(L')^R = 0$ since $\mathcal{P} = 0$ since $\text{deg } \mathcal{P} \geq 2$.
 Thus $L_z^k = \sum \binom{k}{i} \mathcal{P}^i L^{k-i}$. In exercise, you



Came up w/ basis for \overline{Bw}^{-k} in terms of \overline{Bw}^{-k+1} and \overline{Bw}^{-k-1} (maybe)

(3)

Regardless, choose $\{e_i\}$ of \overline{Bw}^{-k+1} projecting to ONB of \overline{Bw}^{-k-1}

$\overline{Bw}^{-k+1} \xrightarrow{A} \overline{Bw}^{-k} \xrightarrow{C} \overline{Bw}^{-k-1}$

get $\begin{bmatrix} \alpha_i \\ b \\ p \end{bmatrix} \begin{bmatrix} B_i \end{bmatrix} \begin{bmatrix} L_{i,j} \end{bmatrix}$

Exercise Compute $\langle v, P^{k+1} L' w \rangle$ for efts of this form and determine that the signature is equal to the signature of $(,)$ on P^{-k+1} .
 Actual signature of $L_{\mathbb{Z}}^k$ must agree for $\mathbb{Z} \gg 0$. \square

So if we can show $hL(\omega, s)_{\mathbb{Z}}$ for $\mathbb{Z} \gg 0$ then we get $HR(\omega, s)_0$.

(Proof for $\mathbb{Z}=0, \mathbb{Z}>0$ are separate). We will use the Prop from last time (as is weak let / strong let)

Need a deg $\neq 1$ map to low terms. Comes from a Rouquier complex

First, a key observation:

$$L_{\mathbb{Z}}^k = \Delta \left(\begin{array}{|c|} \hline \text{|||||} \\ \hline \end{array} \right) + \left(\begin{array}{|c|} \hline \text{|||||} \\ \hline \end{array} \right) \Delta$$

$$= \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array}$$

$$= \dots = \sum z_i \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \quad \square$$

Claim (Exercise): $z_i > 0$

So $L_{\mathbb{Z}}^k$ factors as $\bigoplus_i \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} : BS(\omega) \rightarrow \bigoplus_i BS_i(\omega)$ ↖ leave out $u^{\mathbb{Z}}$ term

compare with $\bigoplus_i \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array}$ in other direction
 adjust maps!

OR just use $\bigoplus_i \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} \begin{array}{|c|} \hline \text{||} \\ \hline \end{array}$ first differential in a Rouquier complex
 let rescale form on $BS_i(\omega)$ by $z_i > 0$.

$BS_i(\omega)$ doesn't have HR though ... it's not semismall. Need to somehow ignore the non-semismall part ... homological alg of Rouquier complexes.

Now, start Rouquier complexes as in lecture 4.6 for Aarhus