Ben Elias — Chennai Lectures VI — 1/2014 — (1)

Outline of proof of Soergel conjecture / recap:

$S(w) := [B_w] = \underline{H}_w$. If true, then $\exists !$ non-zero invariant $\langle , \rangle$ on B_w, up to scalar, and it is nondegenerate.

$\langle , \rangle_{BS(\underline{w})}|_{B_w}$ is non-zero, so nondeg.

$hL(\underline{w}) :=$ $\overline{B_w}$ with $\langle , \rangle_{BS(\underline{w})}|_{B_w}$ and Lefschetz operator given by left mult by ρ with $\partial_s(\rho) > 0 \ \forall s$ satisfies hL.

rex

$hL(w) := hL(\underline{w}) \ \forall \underline{w}$ rex.

$HR(\underline{w}) :=$ — // — HR. Remark: $\langle , \rangle_{BS(\underline{w})}$ already normalized s.t. $(c_{bot}, c_{bot})_\rho^{-l(w)} > 0$.

As discussed, for induction we want to investigate the semismall defect $B_w B_s$, when $ws > w$.

$hL(\underline{w}, s) := \overline{B_w B_s}$ w/ $\langle , \rangle_{BS(\underline{ws})}|_{B_w B_s} \cdots$

$HR(\underline{w}, s) := \ldots$

There will be yet more, but enough for now.

Assume $S(y)$ $hL(y)$ $HR(y)$ $\forall y < ws$ (including $y = w$)

Then $\underline{H}_w \underline{H}_s = \underline{H}_{ws} + \sum \mu(w,s,y) \underline{H}_y$ and $\mu(w,s,y) = \dim \mathrm{Hom}^0(B_w B_s, B_y) = \dim \mathrm{Hom}^0(B_y, B_w B_s)$ and no negative degree maps.

The pairing $\mathrm{Hom}^0(B_y, B_w B_s) \times \mathrm{Hom}^0(B_w B_s, B_y) \longrightarrow \mathrm{End}^0(B_y) = \mathbb{R}$ is the LI Pairing

Both B_y and $B_w B_s$ have non-deg forms $\langle , \rangle_{B_y}$ $\langle , \rangle_{B_w B_s}$ ← see exercises

so given $\varphi \in \mathrm{Hom}(B_y, B_w B_s)$, its adjoint $\varphi^* : B_w B_s \to B_y$ satisfies $\langle \varphi(b), b' \rangle_{B_w B_s} = \langle b, \varphi^*(b') \rangle_{B_y}$

This gives identification $\mathrm{Hom}^0(B_w B_s, B_y) \cong \mathrm{Hom}^0(B_y, B_w B_s)$ allowing transfer of LIP to LIF

$$(\varphi, \psi)_y^{w,s} = \text{coeff of } \mathbb{1} \text{ in } \psi^* \circ \varphi .$$

Embedding Thm: $\mathrm{Hom}(B_y, B_w B_s) \longrightarrow \overline{B_w B_s}$ has image inside $P_\rho^{-l(y)}$, is injective,
$\varphi \longmapsto \overline{\varphi(c_{bot})}$ is isometry up to pos. scalar.

Pf: a) Clearly $\rho^{l(y)+1} c_{bot} = 0$ in $\overline{B_y}$, for degree reasons. So $\rho^{l(y)+1} \overline{\varphi(c_{bot})} = 0$.
Lies in degree $\deg c_{bot} = -l(y)$.

~~b) $\varphi = \sum$ coeff, and $\deg \varphi = 0$, so~~

b) Any map $BS(\underline{y}) \to BS(\underline{ws})$ is $\sum \dfrac{\overline{\Pi}^{\,ws}_{z}}{\underline{LL}_{\underline{y}}}\,\text{coeff}$ ← ignoring choice of rex.

$\quad\cup\oplus \qquad \cup\oplus$

Actually, thinking as factoring thru B_z.

But we're interested in $B_y \to B_wB_s$.

For B_y, By semismall, $\exists$ no deg <0 maps to B_z
$\exists$ deg 0 iff $y=z$.

By semismall B_wB_s, $\exists$ no deg <0 maps from B_z.

So $\begin{array}{|c|} e_w \\ \Pi^2 \\ LL_1 \\ e_y \end{array}\ {}^{z}_{\text{coeff}} = 0$ unless $y=z$, $LL_1 = \mathbb{1}$, deg $\Pi^2 = 0$, coeff = scalar.

We've seen that $\overline{\Pi}^{\,ws}_{z}\,(c_{bot})$ form a basis, so $\overline{\varphi(c_{bot})} = 0 \implies c_{bot} = 0$.

c) We know $\langle c_{bot}, c_{top}\rangle_{B_y} = 1$ and $\langle c_{bot}, \rho^{\ell(y)} c_{bot}\rangle = N > 0$. Thus

$$(\varphi,\psi)^{w,s}_{y} = \text{coeff of } \mathbb{1} \text{ in } \psi^*\circ\varphi = \langle \psi^*\circ\varphi(c_{bot}), c_{top}\rangle = \frac{1}{N}\langle \psi^*\circ\varphi(c_{bot}), \rho^{\ell(y)} c_{bot}\rangle$$

$$\|$$

$$\frac{1}{N}\left(\overline{\varphi(c_{bot})}, \overline{\psi(c_{bot})}\right)^{-\ell(y)}_{\rho} = \frac{1}{N}\langle \varphi(c_{bot}), \psi(\rho^{\ell(y)} c_{bot})\rangle \quad \blacksquare$$

Cor: $HR(\underline{w},s) \implies LIF$ is non-deg $\forall y$ $\implies B_y$ has correct multiplicity in B_wB_s $\forall y$
$\implies [B_{ws}] = \underline{H}_{ws}$
i.e. $S(ws)$.

Also, $B_{ws} \overset{\oplus}{\subset} B_wB_s$ is preserved by ρ, restriction of HR has HR (so long as restriction of $\langle\,,\rangle$ is nondeg) $\implies hL(\underline{ws}), HR(\underline{ws})$.
which it is by $S(ws)$

So ETS $\quad S(<ws), hL(<ws), HR(<ws) \implies HR(\underline{w},s)$.

We use a deformation argument. Let $L_\zeta \in B_wB_s$ be given by $\rho\,\boxed{w}\,| + \boxed{w}\,\zeta\rho|$.
$\zeta\in\mathbb{R}$.

Then $hL(w,s)_\zeta :=$ same but for L_ζ, not $\rho = L_0$.
$HR(w,s)_\zeta$

Can even define these for case when $ws<w$! Exercise: $hL(w) \implies hL(w,s)_\zeta$ when $ws<w$ and $\zeta\neq 0$.

Prop: $HR(w) \implies HR(w,s)_\zeta$ for $\zeta \gg 0$. Pf:
($ws>w$ or $ws<w$.)

$L_\zeta = \rho + \zeta L'$, so L_ζ^k has binomial expansion. However, $(L')^2 = 0$ since $f| = |\boxed{sf}| + |\boxed{\partial f}|$, deg ≥ 2
Thus $L_\zeta^k = \rho^k + \zeta k \rho^{k-1} L'$. In exercise, you

Came up w/ basis for $\overline{B_{\underline{w}} B_s}^{-k}$ in terms of $\overline{B_{\underline{w}}}^{-k+1}$ and $\overline{B_{\underline{w}}}^{-k-1}$ (maybe)

~~choose basis of primitives &~~

Regardless, choose ~~elts~~ α_i of $\overline{B_{\underline{w}}}^{-k-1}$ and projecting to ONB of $\overline{B_{\underline{w}}}^{-k-1}$

$\xrightarrow{\ \ \beta_i\ \ } B_{\underline{w}}^{-k+1}$ $\qquad P^{-k+1} \subset \overline{B_{\underline{w}}}^{-k+1}$

get $\qquad \boxed{\alpha_i}\,{}^{b}_{\phi} \qquad \boxed{\beta_i}\,| \qquad \boxed{L\alpha_i}\,|$

exercise: Compute $\langle v, \rho^{k+1} L' w \rangle$ for elts of this form and determine that the signature is equal

to the signature of $(\ ,\ )^{-k+1}$ on P^{-k+1}. ∎

Actual signature of L_ζ^k must agree, for $\zeta \gg 0$.

So if we can show $hL(\underline{w}, s)_\zeta$ for $\zeta \geq 0$ then we get $HR(\underline{w}, s)_0$.

(Proof for $\zeta = 0$, $\zeta > 0$ are separate). We will use the Prop from last time (as is weak Lef / hard Lef)

Need a deg +1 map to lower terms. Comes from a Rouquier complex

First, a key observation:

$$L_\zeta = \rho\, |||||| + |||||\, \zeta\rho |$$

$$= \partial_{s}(\rho)\, {}^{b}_{\phi}|||||| + |\, s(\rho)\, |||| \zeta\rho |$$

$$= \ldots = \sum z_i\, ||\, {}^{b}_{\phi}\, |||| + ||||||| \boxed{R_+}$$

Claim (Exercise): $z_i > 0$

So L_ζ factors as $\bigoplus \sqrt{z_i}\, |||\, \phi\, |||\ :\ BS(\underline{ws}) \longrightarrow \bigoplus_i BS_{\hat{i}}(\underline{ws})$ ← leave out i^{th} term

composed with $\bigoplus \sqrt{z_i}\, |||\, {}^{b}|||$ in other direction

adjoint maps!

OR just use $\bigoplus ||||\, \phi\, |||$ ← first differential in a Rouquier complex but rescale form on $BS_{\hat{i}}(\underline{ws})$ by $z_i > 0$.

$BS_{\hat{i}}(\underline{ws})$ doesn't have HR though ... it's not semismall. Need to somehow

ignore the non-semismall part ... homological alg of Rouquier complexes.

Now, start Rouquier complexes as in lecture 4.L from Aarhus