Ben Elias — Chennai Lecture V — 1/2014 — (1)

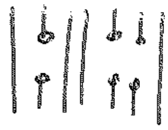
Elements of BS Bim: For a while we've been discussing morphisms b/w BS bims and ignoring that they are actually bimodules. Let's return to bimodules, but study elements using morphisms!

Recall: $B_s = R \otimes_{R^s} R(1)$ has basis as right R-mod $c_1 = 1 \otimes 1$, $\deg c_1 = -1$; $c_s = \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, $\deg c_s = +1$

Now $c_s =$  (c_1) $\quad f c_s = c_s f \quad f c_1 = c_1 s(f) + c_s \partial_s(f)$

If $\underline{\varepsilon}$ a 0,1 sequence $c_{\underline{\varepsilon}} = c_{s_1}^{\varepsilon_1} c_{s_2}^{\varepsilon_2} \ldots c_{s_d}^{\varepsilon_d} \in B_{s_1} \ldots B_{s_d}$

$c_{\underline{\varepsilon}}$ form a basis of $BS(\underline{w})$ as right R-mod

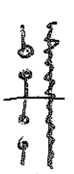
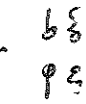
 $=$  $(c_{0000000})$ $\quad c_{bot} = c_{0000000} = 1 \otimes 1 \otimes \ldots \otimes 1$

$c_{1011110}$

Exercise: $f c_{\underline{\varepsilon}} = \sum c_{\underline{\varepsilon}'} g_{\underline{\varepsilon}'}$

Cor: Every elt of $BS(\underline{w})$ is $\varphi(c_{bot})$ for some $\varphi \in End(BS(\underline{w}))$ (obvious, mult by R in any spot)

NOW, $BS(\underline{w}) = R \otimes \ldots \otimes R$ also has a commutative ring structure! (termwise mult)

Ex: $c_{10} c_{10} \overset{\text{why?}}{=}$  $(c_{bot}) =$  $t(\alpha_s) +$  $a_{ts} = c_{10}\, t(\alpha_s) + c_{11}\, a_{ts}$.

These broken line sequences form a commutative subring inside $End(BS(\underline{w}))$, under mult by R.

$\varphi(c_{bot}) \psi(c_{bot}) = \varphi \circ \psi (c_{bot})$ inside this subring, but not in general !!!!

Rmk: Look at Cor. We have a basis for $End(BS(\underline{w}))$! [diagram]

Claim: [diagram: LL$_{\underline{w},\underline{e}}$] $(c_{bot}) = 0$ if $\underline{e}$ has any D's

Recall: For each $x \leq \underline{w}$ $\exists!$ $\underline{e}$ for x w/o D's, called can_x.

Claim: [diagram: LL$_{\underline{w},can}$] $(c_{bot}) = c_{bot}$, with c_{bot} in $BS(\underline{w})$ on the left and in $BS(\underline{x})$ on the right

So have a different basis of $BS(\underline{w})$: [diagram: LL^{I}] (c_{bot}) for $I \subset \underline{w}$. ← Adapted to some things, but not mult! $\varphi(c_{bot})\psi(c_{bot}) \neq \varphi\psi(c_{bot})$

CoB? Mult rules??

→ [diagram] injective

Global intersection form:

$C_{top} = C_{1111111} = $ [diagram] (C_{bot})

$\langle a,b \rangle = $ coeff of C_{top} in ab — Global Int Form (value in R)

Chosen right, not left !!

Ex: $BS(st)$

	C_{bot}	C_{01}	C_{10}	C_{top}
C_{bot}	0	0	0	1
C_{01}	0	0	1	α_t
C_{10}	0	1	$\partial_t(\alpha_s)$	α_s
C_{top}	1	α_t	α_s	$\alpha_s\alpha_t$

Easy upper tri argument $\Rightarrow$ GIF is non-deg.

(What order??)

GIF is also right-invariant: $\langle af, b \rangle = \langle a, bf \rangle = \langle a,b \rangle f$

+ graded $\qquad \langle fa, b \rangle = \langle a, fb \rangle$ ($\neq$ ~)

$\deg \langle a,b \rangle = \deg a + \deg b$.

Exercises have some generalities on duality + invariant forms

Our goal — investigate properties of GIF w.r.t. a signature + Lefschetz operators

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Lefschetz linear algebra — Now it is essential we work over $\mathbb{R}$!

H a gdd V.S. w/ non-deg. bil. form $\langle , \rangle$ graded $H^{-i} \times H^{i} \to R$

$\Longrightarrow \dim H^{i} = \dim H^{-i}$. $\qquad$ **Ex:** $BS(\underline{w})/R_+$ w/ GIF. $/R_+$.

Def: $L : H^{i} \to H^{i+2}$ $\forall i$ is a Lefschetz operator if $\langle v, Lw \rangle^{i} = \langle Lv, w \rangle^{i-2}$

Ex: Left mult by any linear poly. Middle mult too. (right mult would be zero operator)

Def: L induces a form on H^{-i}, the Lefschetz form, via $(v,w)^{i}_{L} = \langle v, L^{i}w \rangle^{i}$

(depends on L!)

Def: L satisfies hard Lefschetz hL if $\forall i \geq 0$

$L^{i} : H^{-i} \to H^{i}$ is injective $\Longleftrightarrow$ it is an isom $\Longleftrightarrow$ $(,)^{-i}_{L}$ is nondegenerate

(0 is trivial!)

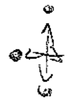
Ex: $BS(st)$ $\qquad L_f = $ left mult by f.

$\partial_s(f) = a$

$\partial_t(f) = b$

$\partial_s(f) = a$, $\quad \partial_t(f - a\alpha_s) = b - a_{ts}a$

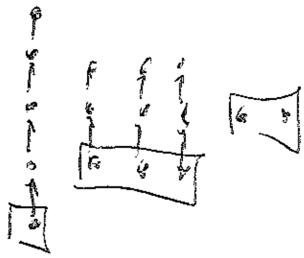
$L^{2}_{f}(C_{bot}) = 2ab - a_{t,s}a^{2}$ satisfies hL if $a \neq 0$, $2b \neq a_{t,s}a$.

Ex: Good example w/o hL:  w/  dual under $\langle , \rangle$

Exercise: L_f on $B_s B_t$ never has hL!! When does $L_f + M_g$ have it? ← middle mult.

Note: $(v,w)_L^{-i} = (Lv, Lw)_L^{-i+2}$ (easy).

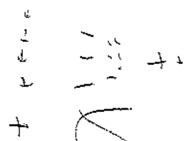
Def: Suppose L has hL. Think of it as an $\mathfrak{sl}_2$ rep.



$P_L^{-i} = \{ v \in H^{-i} \mid L^{i+1} v = 0 \}$ "lowest wt"

$P_L^{k} = 0$ for $k > 0$.

$H^k = \bigoplus_i L^i P_L^{k-2i}$ is Lefschetz decomp "isotypic"



Claim: $(,)_L^{-k}$ is $\perp$ wrt Lefschetz decomp.

Pf: $(v,w)_L^{-k} = (L^i v', L^j w') = \langle v', \underbrace{L^{i+j+k} w'}_{=0} \rangle = 0$. Example.

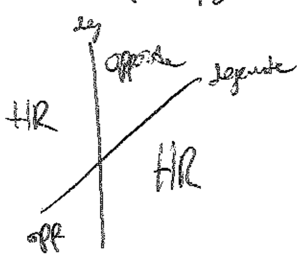
Def: Suppose L has hL. L has the <u>Hodge-Riemann bilinear relations w/ std sgn</u> and H is even or odd. (HR) iff $(,)_L^{-i}$ is <u>alternating definite</u> on primitives

$\Longleftrightarrow$ signature determined via certain formula from graded rank.

Ex: $BS(st)$ again. $(1\xi, 1\xi) = 2ab - a_{t,s} a^2 > 0$.

$\gamma \cdot \xi + \delta \cdot \xi \in \operatorname{Ker} L \implies b\gamma + a\delta = 0$ so gen by $\gamma = a, \delta = -b$.

$(a \cdot \xi - b \cdot \xi, a \cdot \xi - b \cdot \xi) = \langle \; , \; \rangle = -2ab + a_{t,s} a^2 < 0$.



Claim: Given a continuous family L_ζ of operators <u>which all have hL</u>, if any has HR then all have HR.

Pf: $(,)_{L_\zeta}^{-i}$ is a family of <u>nondeg</u> forms, and signature can not change.

Exercise: L_ρ on $B_sB_tB_s$ when has HR? hL?

Def: A homom. of Lefschetz spaces of degree d $(H, L, \langle , \rangle)$ is a map $V^i \xrightarrow{\sigma} W^{i+d}$ s.t. $\sigma L_V = L_W \sigma$.

It's a weird notion.

Ex: $(B_{s(\rho)} \longrightarrow (B_sB_s, L_\rho)$ Y degree -1

Claim: If σ has degree $-d<0$, then $(,)^k_{L_W}$ restricts to zero on the image of σ.

[illegible]

> Ex: If $v \in V$ primitive of degree $-i$
>
> $L^{i+1}v = 0 \implies L^{i+d}(\sigma v) = 0$
>
> $\sigma(v) \in W^{-i-d}$ and
>
> so $(v, w)^{-i-d}_{L_W} = 0$.

Exercise: $(B_s, L_\rho) \longrightarrow (B_sB_s, L_\rho)$ $\frac{1}{2}Y$ degree $+1$

restriction is nonzero!!

One shouldn't expect HR on even hL for $BS(\underline{w})$ only when it has no shifts

$B_sB_tB_s = B_{sts} \oplus B_s$ OK $\qquad B_sB_s = B_s(1) \oplus B_s(-1)$ not OK

OUR MAIN GOAL:

$\bar{B} = B \otimes_R \mathbb{R}$ right quotient, finite dim. If $B \cong \mathbb{D}B$ then $\dim B^i = \dim B^{-i}$ and $\exists \langle , \rangle$ nondegen.

L_ρ is a Lefschetz operator

Choose $\rho \in \mathfrak{h}^*$ st $\partial_s(\rho) > 0$ $\forall s$.

Then $(\bar{B}_w, L_\rho)$ has HR!! More specifically, $B_w \overset{\oplus}{\subset} BS(\underline{w})$ ← rex

and $BS(\underline{w})$ has form $\langle , \rangle_{GIF}$, restricts to B_w having $\langle , \rangle_{GIF}$. HR wrt this form.

MAIN GOAL ⟹ Soergel Conj: Next time, but idea is to show that some LIF embeds into GIF into primitive subspace, ergo definite, ergo nondegenerate.

In fact, LIF is ± definite!

Key Step in proof of main goal: Induction. "Weak Lefschetz"

Spose σ a map of degree 1 from $V \to W$ and W has HR.

Spose $(\sigma v, \sigma w) = (v, v')$. What can you say about V?

Spose σ injective from negative degrees.

$v \in P_L^{-i}$. $0 \neq \sigma v = w_0 + Lw_1 + L^2w_2 + \ldots \in W^{-i+1}$ so $L^i v \neq 0$ if any $w_i \neq 0$ for $i > 0$

$L^{i}\sigma v = L^{i}w_0 + L^{i+1}w_1 + \ldots$ If $w_i = 0$ $\forall i > 0$, σv is primitive.

Then $(v,v) = (\sigma v, \sigma v)^{-(i+1)}$ $\neq 0 \Longrightarrow L^i v \neq 0$. So V has hL!!
$\langle v, L^i v \rangle$

<u>Prop:</u> $V \xrightarrow{\sigma} W$ of degree 1, $\sigma L = L\sigma$, $(\sigma v, \sigma v')_L = (v,v')_L$

and σ injective from negative degrees. Then HR for $W \Longrightarrow$ hL for V.

(If also σ preserves primitives of neg degree, then HR for $W \Longrightarrow$ HR for V except in degree 0.)

Where did this idea come from? Geometric background

X sm. proj vty, $\mathbb{C}$ dim $= n$. $H^*(X)$ has Poincaré Duality, so $\dim H^{n-i} = \dim H^{n+i}$

Intersection pairing b/w H^{n-i} and H^{n+i}

$\mathcal{L} \downarrow X$ ample line bundle, $c_1(\mathcal{L}) \in H^2(X)$ gives Lefschetz form (H^2 commutes w/ H^*)

<u>Thm</u> (hard Lefschetz): $c_1(\mathcal{L})$ has hL and HR.

(assume $H^*(X)$ is even, otherwise need more care)

<u>Idea of proof:</u> Choose generic section in $\Gamma(\mathcal{L})$, gives a hyperplane $Y \overset{i}{\subset} X$ of codim 1. By induction, $H^*(Y)$ has hL, HR.

$H^*(Y) \underset{i^*}{\overset{i_*}{\rightleftarrows}} H^*(X)$ composition is $c_1(\mathcal{L})$, so can show satisfies Prop.

<u>Weak Lefschetz Thm:</u> i^* is injective from negative degrees, isom to neg degrees

i_* is surj to pos degrees, isom from pos degrees.

$\underset{\text{Prop}}{\Longrightarrow}$ hL for $H^*(X)$, HR outside degree 0.

All that remains is a signature argument for $H^0(X)$.