Ben Elias — Chennai Lectures Part IV — 1/2014

Recap: BSBim is an alg. defined cat of bimodules over R. We've described it diagrammatically allowing its study w/o actually worrying about the bimodules themselves.

Def: Let D be monoidal cat w/ 1-mor generators $s \in S$

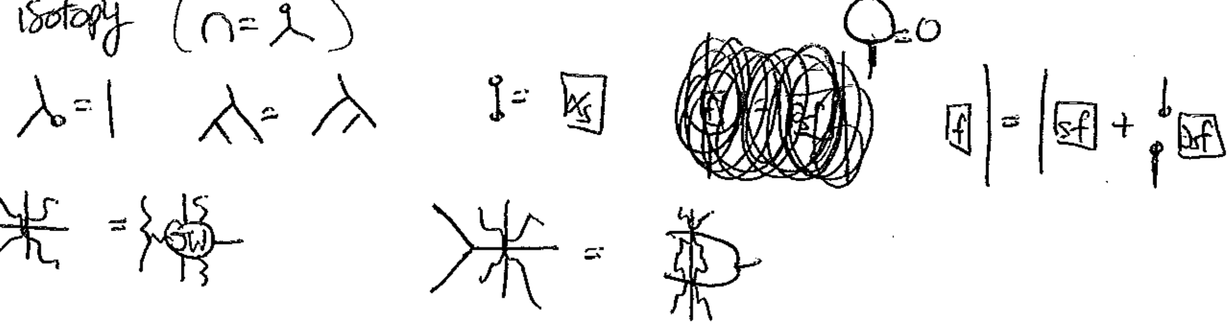
2-mor gens:

$\boxed{f} \in R$

relations: isotopy ($\cap = \lambda$)

1-color:

2-color:



3 colors: Zams

Def: Let $F: D \to$ BSBim send $s \mapsto B_s$, (dot) $\mapsto$ mult etc

(4-valent vertex) $\mapsto$ hard to describe!

Thm: (E-W) F is well-defined and is an equiv of cats. Thus $Kar(F): Kar(D) \to$ SBim $\cong Kar(BSBim)$

Reminder: Given a cat $\mathcal{C}$, $M \in \mathcal{C}$ and $e \in End(M)$ with $e^2 = e$, can formally add "Im e" as an object.

$Hom(Y, Im\, e) = e\, Hom(Y, M)$

$Hom(Im\, e, Z) = Hom(M, Z)\, e$.

Rmk: If Im e was already an object, X, then $X \cong Im\, e$.

Doing this for all idempotents, get $Kar(\mathcal{C})$.

Today we fill in three related gaps —

1. Finding bases for Hom spaces (now that have a language to describe them)
2. Intersection forms — helping to understand Kar(D)
3. Tools for discussing elements of BS Bimodules morphism-theoretically — allows return to algebra

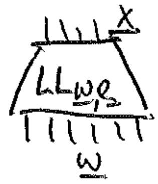
Finding Bases: Light Leaves (Libedinsky) Recall $\underline{H}_{\underline{w}} = \sum_{\underline{e} \subset \underline{w}} v^{d(\underline{e})} H_{\underline{w}^{\underline{e}}}$.

and SH Formula $\operatorname{rank} \operatorname{Hom}(BS(\underline{w}), BS(\underline{y})) = \langle \underline{H}_{\underline{w}}, \underline{H}_{\underline{y}} \rangle$

$$= \sum_{x} \sum_{\substack{\underline{e} \subset \underline{w} \\ \underline{w}^{\underline{e}} = x}} \sum_{\substack{\underline{f} \subset \underline{y} \\ \underline{y}^{\underline{f}} = x}} v^{d(\underline{e}) + d(\underline{f})}$$

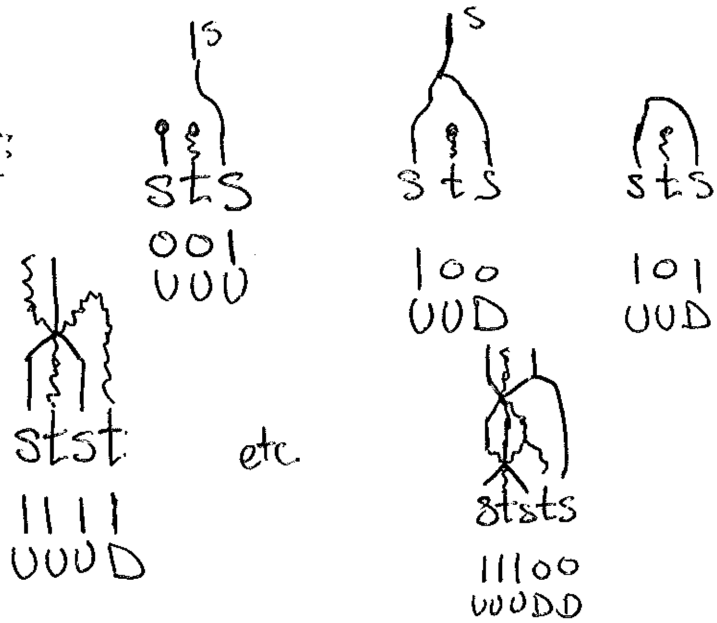
So let's find a basis, indexed by $x, \underline{e}, \underline{f}$. Really, 2 halves of each morphism $x, \underline{e}$ and $x, \underline{f}$

Choose + fix arbitrary rex for x, $\underline{x}$. We construct

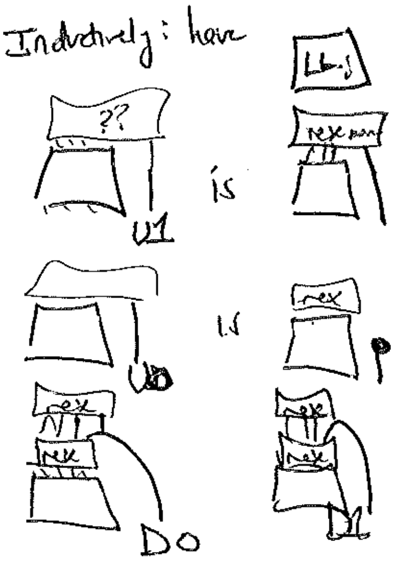


Not canonical, depends on some choices.

Example first:

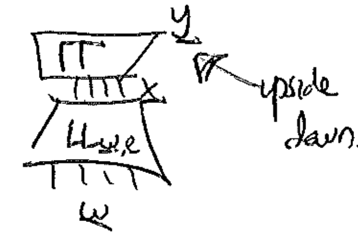


Inductively: have



Well defined up to choices of rex moves.

Def: $\underline{e} \subset \underline{w}$ $\underline{f} \subset \underline{y}$ $\underline{w}^{\underline{e}} = \underline{y}^{\underline{f}}$ then $\mathbb{LL}_{\underline{e},\underline{f}}$ is



Thm: $\{\mathbb{LL}_{\underline{e},\underline{f}}\}$ form a basis for $\operatorname{Hom}(\underline{w}, \underline{y})$ as right R-mod

Much practice needed! Exercises

Consequences: (1) I an ideal in the Bruhat order, like $\leq w$, $< w$,

$\mathcal{D}_I$ = morphisms factoring thru rexes for $v \in I$

$= \operatorname{span} \{\mathbb{LL}_{\underline{e},\underline{f}}\}_{\underline{w}^{\underline{e}} = \underline{y}^{\underline{f}} = v \in I}$

this is an ideal in $\mathcal{D}$.

When w is understood, "$\mathcal{D}_{<w}$ is "lower terms"

② $\underline{w}, \underline{w}'$ two rexes, β, β' two rex moves 

then $\beta - \beta' \in \mathcal{D}_{<w}$.

So LL is well defined <u>modulo lower terms</u>!

③ $\mathrm{End}(\underline{w}) = \mathbb{1} + \text{lower terms}$ $\rightsquigarrow$ classification of indecomposable objects in $\mathrm{Kar}(\mathcal{D})$.
rex

④ $\mathcal{D}$ is an <u>object adapted cellular category</u>:

Equipped with $\Lambda \subset \mathrm{Ob}(\mathcal{D})$ (a rex for each $w \in W$)
$\leq$ p.o. on Λ (Bruhat order)
i antiinvolution (flipping diagram upside down)

For each $X \in \mathrm{Ob}(\mathcal{D})$ $\lambda \in \Lambda$ a finite set $E(X, \lambda)$ (this $E(\underline{w}, x) = \{\underline{e} \subset \underline{w}, \underline{w}^{\underline{e}} = x\}$)
$\updownarrow i$
$M(\lambda, X)$ (same)

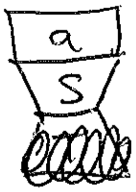
and a morphism $C_T \in \mathrm{Hom}(X, \lambda)$ s.t. $i(C_S) = C_{i(S)}$
$\forall T \in E(X, \lambda)$
$S \in M(\lambda, Y)$ $C_S \in \mathrm{Hom}(\lambda, Y)$

($C_{\underline{e}} = LL_{\underline{w}, \underline{e}}$)

satisfying ① $\{C^{\lambda}_{S,T}\}_{\lambda \in \Lambda, S, T}$ is a basis for $\mathrm{Hom}(X, Y)$
$\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; = C_S \circ C_T$ ($\mathbb{LL}$)

② $a C_S = \sum l(a, S, S') C_{S'} + \text{lower terms}$ ← span of $C^{\mu}_{U,V}$, $\mu < \lambda$.

$\implies a C_{S,T} = \sum l(a, S, S') C_{S', T} + \text{l.t.}$ ← indep of T.
← not indep of T.

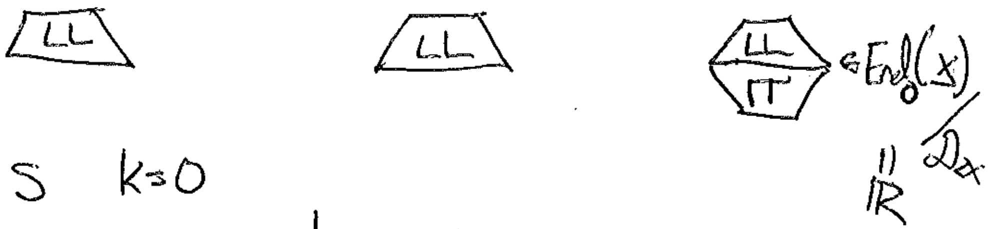
Idea  = some term  +  lower terms (difficult)

 =  +  ← lower terms

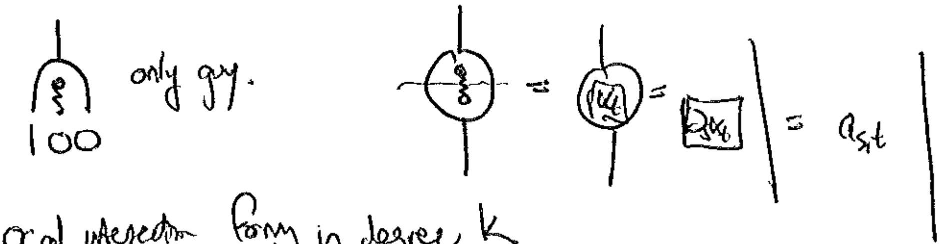
Exercise: Walk thru proof that indecomposables in a obj. ad. cell. cat. are classified by Λ. – $\lambda \in \Lambda$ has ! ind. idempotent e, $e = 1$ modulo l.t.

Intersection form: Fix $\underline{w}$, x, $k \geq 0$. Consider pairing

$$\{e \subset \underline{w},\ \underline{w}^{e} = x,\ d(e) = k\} \times \{e \subset \underline{w},\ \underline{w}^{e} = x,\ d(e) = -k\} \longrightarrow \mathbb{R}$$

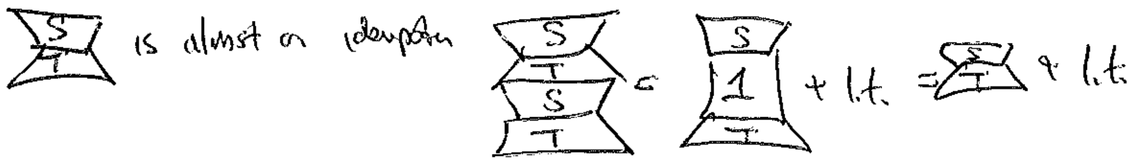


Ex: $\underline{w} = sts$ $x = s$ $k = 0$

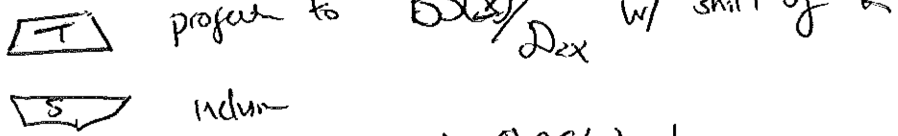


This is the local intersection form in degree k

Suppose S, T pair to 1.



using std arguments, can complete to an idempotent,



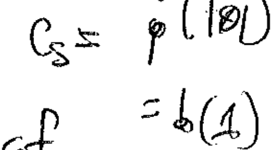
so $B_x(k) \overset{\oplus}{\subset} BS(\underline{w})$

flipping upside down, $B_x(-k) \overset{\oplus}{\subset} BS(\underline{w})$ too.

rank of LIF gives multiplicity of summand.

Elements of BS Bundles:

Recall! $B_s = R \otimes_{R^s} R$ has basis $c_1 = 1 \otimes 1$ $c_s = \frac{\alpha_s \otimes 1 + 1 \otimes \alpha_s}{2}$ $f c_s = c_s f$.



So given $\underline{\varepsilon}$ a 0,1 sequence, $C_{\underline{\varepsilon}} = C_{s_1} C_{s_2} \ldots C_{s_d} \in B_{s_1} B_{s_2} \ldots B_{s_d}$

Claim! $\{C_{\underline{\varepsilon}}\}$ form a basis for $BS(\underline{w})$ as a right R-mod.

$C_{\underline{\varepsilon}} = \, |\,\dot{}\,|\,|\,|\,\dot{}\,\dot{}\,| \, (C_{bot})$ $\quad C_{bot} = C_0 C_0 C_0 \ldots C_0 = |\otimes|\otimes|\otimes \ldots \otimes|$

Cor! Every elt of $BS(\underline{w})$ is $\varphi(C_{bot})$ for $\varphi \in End(BS(\underline{w}))$.

But we have a basis of $End(BS(\underline{w}))$! UU/LL.

Claim! $LL_{\underline{w},\underline{e}}(C_{bot}) = 0$ if $\underline{e}$ has any D's.

Recall from exercise: for each $x \leq \underline{w}$, $\exists!$ $\underline{e} \subset \underline{w}$ with no D's, called can_x. has maximal defect.

Claim! $LL_{\underline{w}, can_x}(C_{bot}) = C_{bot}$
(C_{bot} on the left in $BS(\underline{w})$, on the right in $BS(\underline{x})$)

So every elt of $BS(\underline{w})$ is $UU_f(C_{bot})$ for some f. $\quad C_{top} = 111$ (wwww)

Global intersection form! $\quad BS(\underline{w}) = R \otimes_{R^{s}} R \ldots \otimes_{R^{u}} R(\ )$ is also a ring!

$\langle a,b \rangle$ = coeff of C_{top} in $ab \in R$.

(commutative. But $\varphi(C_{bot})\psi(C_{bot}) \neq \varphi \circ \psi(C_{bot})$)

$C_{top} = C_{s_1} \ldots C_{s_d}$

Ex! $BS(st)$

	C_{bot}	C_{10}	C_{01}	C_{top}
C_{bot}	0	0	0	1
C_{10}	0	$\partial_t(\alpha_s)$	1	α_s
C_{01}	0	1	0	α_t
C_{top}	1	α_s	α_t	$\alpha_s\alpha_t$

easy uppertri argument — GIF is nondegenerate!

GIF is (right) invariant: $\deg\langle a,b\rangle = \deg a + \deg b$ $\quad \langle af, b\rangle = \langle a, bf\rangle = \langle a,b\rangle f$

$\langle fa, b\rangle = \langle a, fb\rangle$ ($\neq$ anything else, can't pull it out)