Lecture 1.4 Part I: Diagrammatics for Categories | We've defined monoidal SBim, & cat SSBim (1)

We use planar diagrams to describe morphisms b/w (Singular) Soergel bimods, but it's no accident!
Planar diagrams are precisely the tool for the job.

Baby case: Linear Diagrams for (1-)categories | You're familiar w/ $P \xleftarrow{g} N \xleftarrow{f} M$

objects fill a pt, morphisms a line. Let's take dual picture. $P \underset{g}{\bullet} N \underset{f}{\bullet} M$

Same data, but has some apparent positioning. $P \xleftarrow{g} N \xleftarrow{\quad f \quad} M$

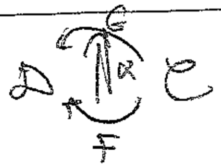
In picture: A (generic) pt is an object

A (stop saying pt) interval is a morphism $[N \underset{f}{\bullet} M]$ from RHS to LHS

Composition $[\bullet\ I\ \bullet]$ identity $[\ M\ ]$ is 1_M

Axioms of a category $\longleftrightarrow$ Diagram (up to linear isotopy) unambiguously represents a morphism.

(We could use positioning to keep track of parens, but no need)

Planar Diags for 2-cats | Old way (2-cat of cats)  New way 

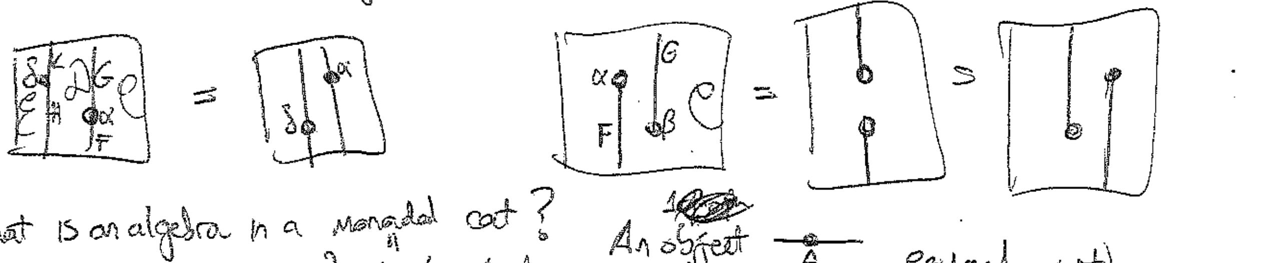
pt $\longleftrightarrow$ object

horiz line $[D \underset{F}{\bullet} C]$ $\longleftrightarrow$ 1-mor, same rules as above, $[\ C\ ] = \mathbb{1}_C$

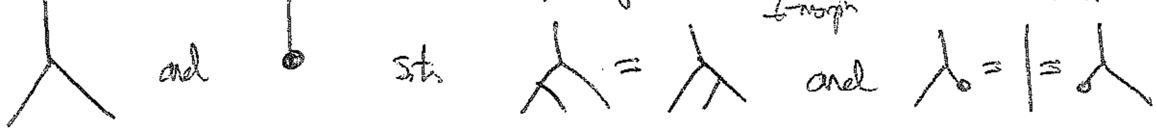
rectangle $\boxed{D \overset{G}{\underset{F}{\bullet}} C}$ $\longleftrightarrow$ 2-mor bottom to top. $\boxed{D \,|_F\, C} = 1_F$ $\boxed{C} = 1_{\mathbb{1}_C}$

compose horiz or vertically.

Axioms of 2-cats $\longleftrightarrow$ Diagram (up to rectilinear isotopy) unambiguously gives a 2-morphism.



Examples: (1) What is an algebra in a monoidal cat ? (= 2-cat w/ one object). An object (= 1-morph) $\underset{A}{\bullet}$ equipped with



② What is a Frob alg object? s.t. algebra, coalg,

and $\cong \cap$ $\cong \cup$ then (can view diagrams up to isotopy!)

Assoc $\Longleftrightarrow$

③ Frobenius extension? $B \uparrow A$ Ind, $A \downarrow B$ Res, $B \otimes_A B \xrightarrow{\mu} B$, ${}_A B_A \xrightarrow{\partial} A$

satisfying (again, isotopy!) + a bit more!

To make graded: deg $= +\ell$, deg $= -\ell$

$\boxed{f}$ for $f \in B, A$.

Exercise I said $B \otimes_A B$ a Frob alg object in B-bimod etc.

$\Updownarrow$

④ When $E \dashv E^\vee$ we have $\mathbb{1} \to EE^\vee$, $E^\vee E \to \mathbb{1}$. If draw $D \uparrow_E C$, $C \downarrow_{E^\vee} D$ then

$\mathcal{D} \underset{F}{\overset{E}{\rightleftarrows}} \mathcal{C}$

and , w/

If biadjoint, also w/

However, if E biadjoint to $E^\vee$, $F \longleftrightarrow F^\vee$, it is possible that $\neq$

If they are equal, $\boxed{\beta}$ is called *cyclic*. Can draw cycle as

Axioms of biadjunction + cyclicity $\longleftrightarrow$ Diagram (up to true isotopy) unambiguously represents a 2-morphism.

(If all 1-morphisms have duals and all 2-morphisms cyclic) Given such a category, you *should* use isotopy classes of diagrams.

Rmk: All 2-morphisms are cyclic when "taking biadjoints" is actually functorial. Common situation in geometry + convolution categories.

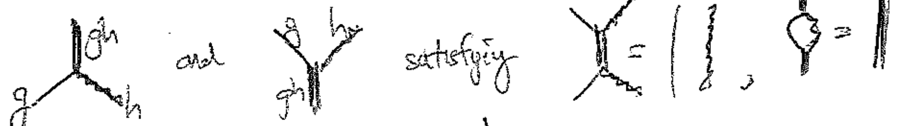
In lectures to come we'll show you how to draw morphisms b/w Bott-Samelson bimodule. You can already draw a lot for $B \otimes B \otimes \ldots \otimes B_s$

Lecture 14 Part II

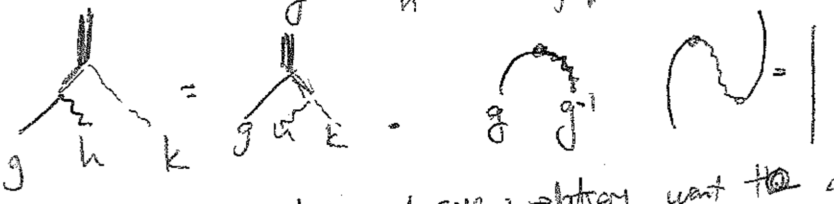
Let's draw another monoidal category.

Def: Let G be a group. The 2-groupoid of G is the monoidal category w/ objects $g \in G$ and $g \otimes h = gh$. Only morphisms are identity maps.

So, for instance there is a map and satisfying



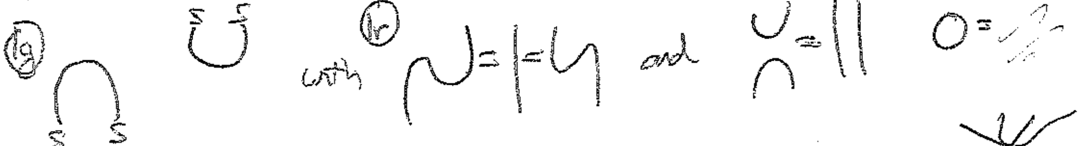
and



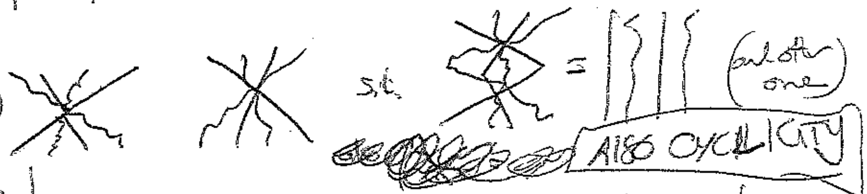
etc.

However, when G has a presentation w/ gens + relations want to abuse that to simplify diagrams.

Ex: $G = (W,S)$ a Coxeter gp. Generated by $s \in S$. Since $s^2 = 1$ have maps (1g) with (1r) and



since (2g) $stst\cdots = tsts\cdots$, $m_{st} < \infty$, have maps (2r) s.t. (and other one)

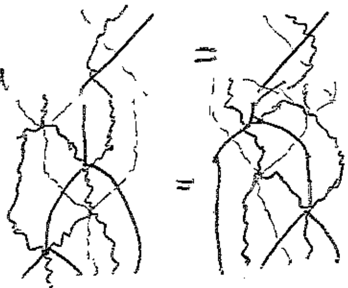


A180 CYCLICITY

Are there any more relations? Sure! but there can be only one, so relation

Spose $m_{st} = m_{tu} = m_{su} = 2$. Two maps $stu \to uts$

Spose •—•—• . Two maps $stsuts = w_0 \to utustu = w_0$



"Zamolodchikov"

Thm (E-W): The following is a diagrammatic presentation for the 2-groupoid of (W,S) for any Coxeter gp

Generators: (1g) (2g) Relations: (1r) (2r) 3r: One such relation for each finite rank 3 Cox subgp

Equality b/w distinct paths in longest element.

Idea: For any w, let Γ_w be the reduced expression graph; vertices - reduced expressions, edges - braid relations. Ex: each row in Zam above.

Any path gives a morphism, any loop better be equal to identity.

Trivial loops: $\supset\!\!\!< \; = ||$; Fact: Nontrivial loops all gen by Zams.

What about non-reduced expressions. Trickier. We proved using topology of Coxeter complexes.