## LECTURE 2.2 — DRAWING SBIM IN RANK ≤ 2

We've defined ⊗-cat BSBim, so should draw morphisms. If you're new to this, think of a diagram as a better encoding of a morphism - much easier than symbols. They have a life of their own too - can define a diag. cat... not for now.

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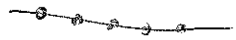
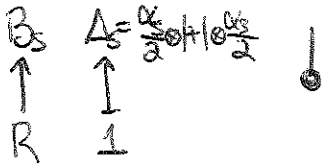
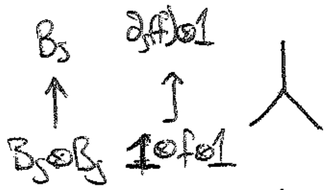
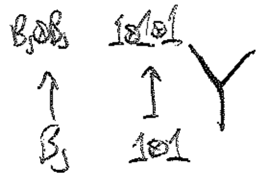
**WARMUP #1:** $S = \{s\}$; BSBim consists of $B_s^{\otimes n}$ 

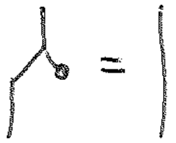
$B_s$ is a frobenius algebra in BSBim so have 4 structure maps:

$\mu: B_s \to R$, $f \otimes g \mapsto fg$  degree $+1$

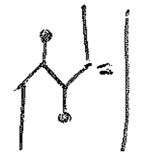
$R \to B_s$, $1 \mapsto \Delta_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$  $+1$

$B_s \otimes B_s \to B_s$, $1 \otimes f \otimes 1 \mapsto \partial_s(f) \otimes 1$  $-1$

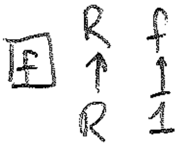
$B_s \to B_s \otimes B_s$, $1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1$  $-1$

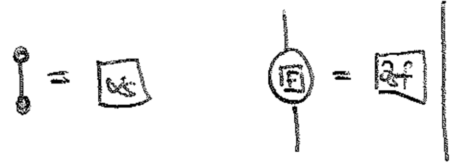
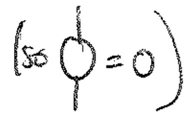
Example computation: 

LHS $1 \otimes 1 \mapsto 1 \otimes \frac{\alpha_s}{2} \otimes 1 + 1 \otimes 1 \otimes \frac{\alpha_s}{2} \mapsto \partial_s(\frac{\alpha_s}{2}) \otimes 1 + \partial_s(1) \otimes \frac{\alpha_s}{2} = 1 \otimes 1$

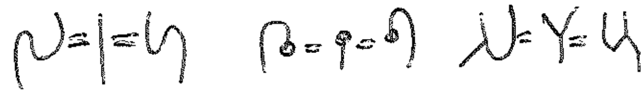
RHS $1 \otimes 1 \mapsto 1 \otimes 1$. 

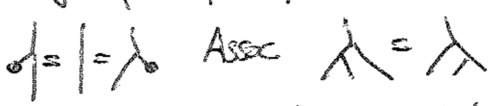
 since $\partial_s(f)$ symmetric.

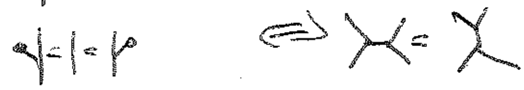
Another generator:  $R \to R$, $1 \mapsto f$

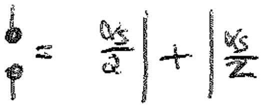
Example:  (so )

**Thm:**  generate all R-bimodule morphisms. Let 

Relations are: Std Frobenius Relations: Isotopy 

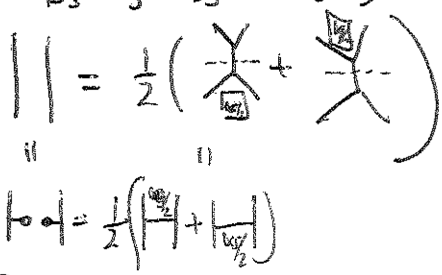
Unit, Assoc 

⇒ Counit ⇐ 

Decomp:  (in general, $\sum a_i | b_i$ for dual bases $\{a_i\}$ $\{b_i\}$ comes from $R \cong R^s \oplus R^s(-2)$)

Eval:  $= \mu(\Delta_s) =$ $\boxed{\alpha_s}$ 

Consequences: $f \otimes 1 = \partial_s(f)\left[\frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}\right] + 1 \otimes s(f)$

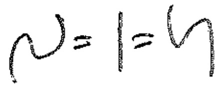
$B_s \otimes B_s \cong B_s(1) \oplus B_s(-1)$  inclusion + projection maps. Check!!

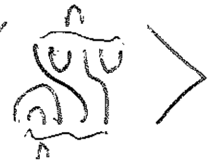
Now $S = \{s, t\}$. Have , but anything else?? How to work with?

Warm up 2: $\mathfrak{sl}_2$ repns. Let  be $V^{\otimes n}$, $V = V_1$ = fund rep, basis $\{e_1, e_2\}$ (2)

Now $V^{\otimes 2} \underset{\oplus}{\supset} \Lambda^2 V = \mathbb{C}$ so $\exists$ inclusion, proj

Let $\cup$: $1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$ ; $\cap$: $e_1 \otimes e_2 \mapsto -1$, $e_2 \otimes e_1 \mapsto +1$, $e_1 \otimes e_1,\ e_2 \otimes e_2 \mapsto 0$. Check $\bigcirc = -2$



Def: $TL_n = \mathbb{C}\langle$  $\rangle$, get category $\mathcal{TL}$ w/ ob $= \mathbb{N}$, Mor $= \langle$ n over m diagrams $\rangle$

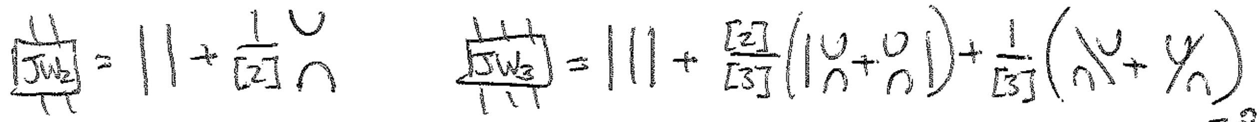
compose = stack, $\bigcirc = -2$.

Rmk: This whole picture has a $q$-defn, giving Rep $U_q(\mathfrak{sl}_2)$, $V = \mathbb{Q}(q)\langle e_1, e_2 \rangle$

$\bigcirc = -(q + q^{-1}) = -[2]$

Thm: $\mathrm{Hom}_{\mathcal{TL}}(m, n) \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V^{\otimes m}, V^{\otimes n})$

Now $V_n \overset{\oplus}{\subset} V^{\otimes n}$ so $\exists$ idemp $V^{\otimes n} \to V_n \to V^{\otimes n}$, $JW_n \in TL_n$ Jones-Wenzl Projector

Ex: 

$[3] = [2]^2 - 1$

Properties: ① $JW_n$ Can be defined w/ coeffs $\frac{[k]}{[n]}$ for $k \le n$ (not quite)

② Nice recursion formulae

③ ! morphism s.t.  and coeff of $||||$ is 1.

} exercises

But $V_n$ not the only summand of $V^{\otimes n}$. $V^{\otimes n} \cong V_k^{\oplus c_{n,k}}$, can deduce from

$$V \otimes V_n = V_{n+1} \oplus V_{n-1} \text{ for } n \ge 1, \quad V \otimes V_0 = V_1$$

so $TL_n$ has all these idempotents too (+ isoms within isotypic component)

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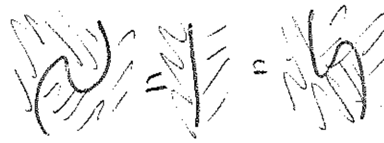
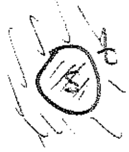
Now for the real deal: $S = \{s, t\}$. Before I start drawing $\mathbb{BSB}im$, gonna do something else:

 is ${}_{R^t}R_{R^s}(1)$ (rank 2 for either side)  is ${}_{R^s}R_{R^t}$. Mixing two Frob exts!

have 

${}_{R^t}R \otimes_{R^s} R_{R^t} \to R^t$, $f \otimes g \mapsto \partial_t(fg)$

$R^t \to R \otimes_{R^s} R$, $1 \mapsto \sum \frac{\alpha_s \otimes 1 + 1 \otimes \alpha_s}{2}$

They satisfy  and  $= \partial_t(\alpha_s) = a_{st} = -2\cos\left(\frac{\pi}{m}\right)$ (3)

$= -(q+q^{-1})$ for $q = \zeta_{2m}$

$= -[2]$

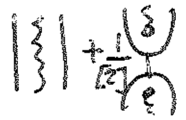
So there is a map from $^{a}TL$ (some 2-colored version of) into $\otimes$s of these guys.

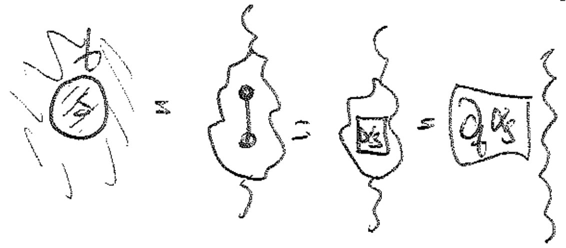
A morphism

$_{R^t}R \otimes_{R^s} R \otimes_{R^t} R_{R^s}$ gives a morphism $R \otimes_{R^t} R \otimes_{R^s} R \otimes_{R^t} R \otimes_{R^s} R = B_t \otimes B_s \otimes B_t \otimes B_s$

So every elt of $^{a}TL$ gives a morphism of $\mathbb{BS}$ Bim! <u>Def Retract</u>.

$JW_2 = || + \frac{1}{[2]} \genfrac{}{}{0pt}{}{\cup}{\cap}$ becomes





Still an idempotent (when $[2] \neq 0$ ie $m_{st} > 2$.)

in fact, $[m_{st}] = 0$ gives $B_{sts} \overset{\oplus}{\subset} B_s B_t B_s$.

Just as $V \otimes V_n \cong V_{n+1} \oplus V_{n-1}$ $V \otimes V_0 \cong V_1$

so here $B_s \otimes B_{\underbrace{tst}_{n+1}} \cong B_{\underbrace{sts}_{n+2}} \oplus B_{\underbrace{st}_{n}}$ $m-2 \geq n \geq 1$ $B_s \otimes B_t = B_{st}$ (see exercises)

Can now explicitly decompose all alternating $\mathbb{BS}$ Bim for $m = \infty$, when all $[m] \neq 0$.

<u>Rmk</u>: This connection $\mathbb{S}\mathrm{Bim}_{\hat{A}_1} \overset{\sim}{\longleftrightarrow} sl_2$-rep is (quantum) geometric Satake!!!

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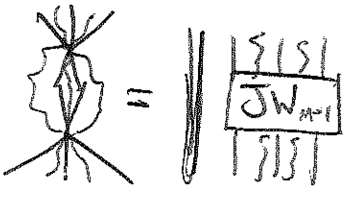
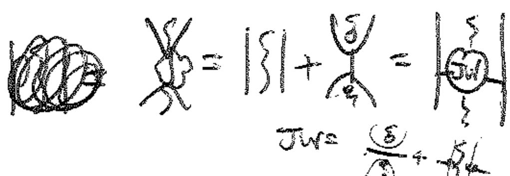
When $m < \infty$, $JW_m$ undefined, no good way to break down $\underbrace{B_s B_t \cdots}_{k}$ for $k \geq m+1$ using $^{a}TL$

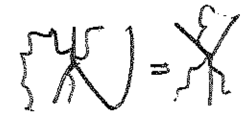
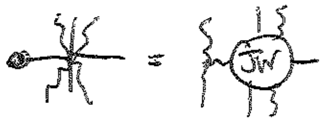
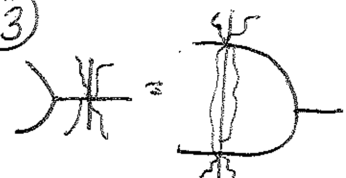
But $JW_{m-1}$ defined, $[m-1] = 1$, its even <u>rotation-invariant</u> in TL!!

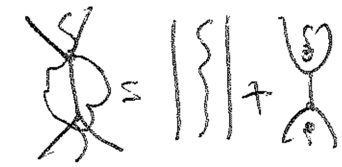
Gives $B_{\underbrace{sts\ldots}_{m}} \overset{\oplus}{\subset} \underbrace{B_s B_t \cdots}_{m}$ and $B_{tst\ldots} \overset{\oplus}{\subset} B_t B_s \cdots$ a common summand.

So $\exists$ deg 0 map $B_s B_t \cdots \rightleftarrows B_t B_s \cdots$ (through $B_{w_0}$) draw as  new generator.

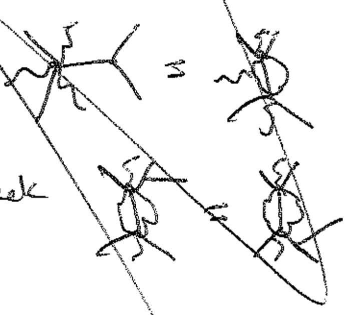
and 

Ex: $m=3$ 

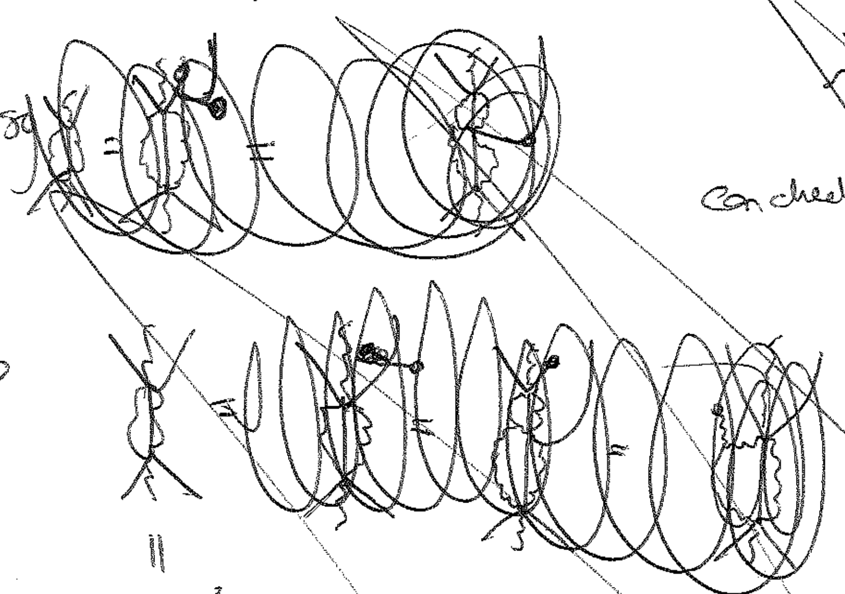
<u>Rels</u>: (1) Isotopy  (2)  (3)  $m=3$ example

Ex: Why 

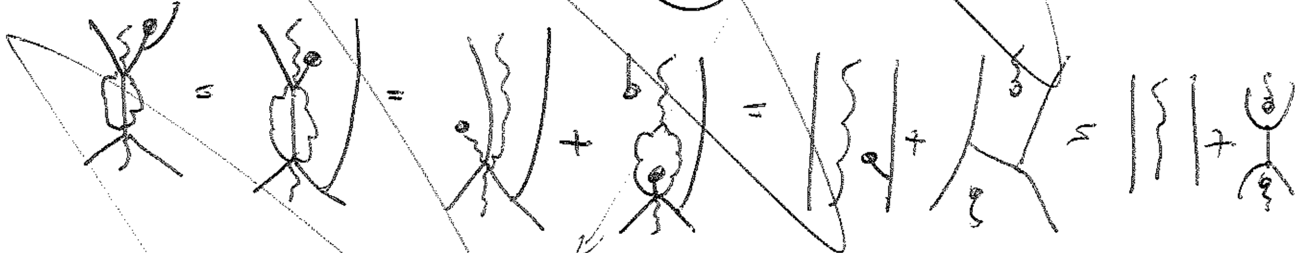
we have 

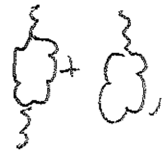


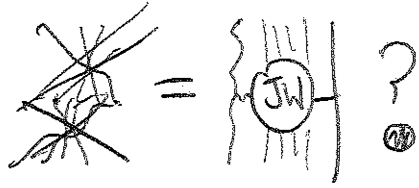
can check

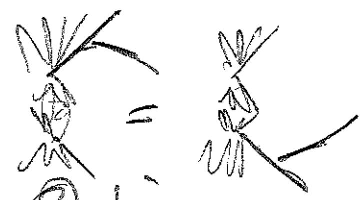


So



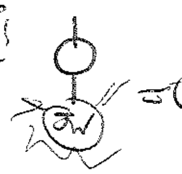
has , both zero

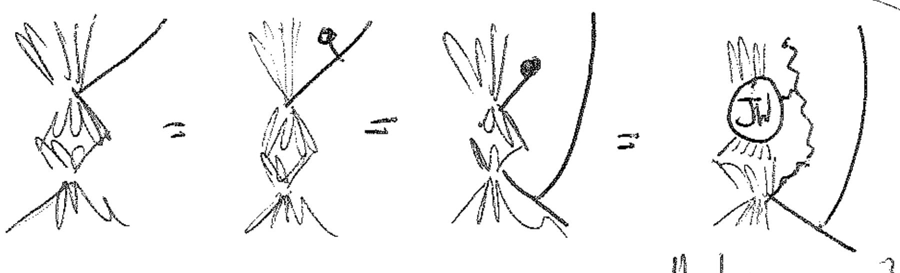
Ex: Why  ?

Claim: 

using (3) twice

Claim: 

Pf: 



all terms except  or have  which kills