Today's goal: Introduce the main players - Soergel bimodules, a catfn of $\mathbb{H}_W$.
Easy to get a handle on since alg. and combinatorial.

§1 Reflection Reps + Polys

Fix (W,S). Define the <u>symmetric Cartan Matrix</u> of (W,S) to be

$$A=\begin{pmatrix} 2 & & a_{st} \\ & 2 & \\ a_{t,s} & \ddots & 2 \end{pmatrix}$$

with $a_{s,t}=a_{t,s}=-2\cos\frac{\pi}{m_{st}}$ (when $m_{st}<\infty$)
(when $m_{st}=\infty$ can use $a_{s,t}=a_{t,s}=\pm 2$ or anything suitably generic)

Let $\mathfrak{h}^*/\mathbb{R}$ have basis $\{\alpha_s\}_{s\in S}$, called <u>simple roots</u>.

$W \circlearrowright \mathfrak{h}^*$ by $s(\alpha_t)=\alpha_t-a_{st}\alpha_s$, so $s(\alpha_s)=-\alpha_s$,
$t\neq s$: $s(\alpha_t)=\alpha_t+2\cos\frac{\pi}{m_{st}}\alpha_s$

Rmk: Many ways to generalize, see exercises.

Def: Let $R=\mathrm{Sym}^*(\mathfrak{h}^*)=\mathbb{R}[\alpha_s] \circlearrowleft W$. Graded, $\deg \alpha_s=2$.
For $I\subset S$, let $R^I=R^{W_I}$ invts under parabolic subgp W_I.

Ex: $W=S_n \circlearrowright \mathbb{R}[x_1,\dots,x_n]/\sum x_i=0$ ← can ignore. $\alpha_i=x_i-x_{i+1}$

$W_I=S_3\times S_1\times S_2\times\dots$ $\quad R^I=\mathbb{R}\left[\begin{matrix}x_1+x_2+x_3\\ x_1x_2+x_1x_3+x_2x_3\\ x_1x_2x_3\end{matrix},\ x_4,\ \begin{matrix}x_5+x_6\\ x_5x_6\end{matrix},\ \dots\right]/\sum x_i=0$

Thm (Chevalley): Suppose W_I is finite $\iff$ I is <u>finitary</u>. Then
R^I is a poly ring (of same transcendence degree $|S|$), generated by alg. indep. polys in various degrees determined by W_I. (Many facts, see Humphreys "Cox. Gps..." $\prod d_i=|W_I|$, $\sum d_i=\dots$)

So the rings R^I aren't so bad. (Compare to other invariant subrings!)
But what is even better is the relationship of R^I to R. Think of this as a souped up Chev Thm.

Thm: $R^I\subset R$ is a <u>Frobenius extension</u>. So is $R^J\subset R^I$ for $I\subset J$.

Def: A (commutative) ring ext. $A\subset B$ is a <u>Frob Ext</u> if it is equipped w/ $\partial: B\to A$, A-linear, and if B is free over A w/ dual bases $\{b_i\}$ and $\{b_i^*\}$ s.t. $\partial(b_ib_j^*)=\delta_{ij}$.
When A,B are graded rings, require dual bases to be homogeneous, and $\deg\partial=-2\ell$ then called Frob Ext of degree ℓ.

Why these, and why "Frobenius" – comes from Frob Reciprocity. Ex: $H \subset G$ $\mathbb{C}[H] \subset \mathbb{C}[G]$ ②

The bimod. ${}_B B_A$ gives functor ${}_B B \otimes_A \circ$: A-mod $\to$ B-mod Induction

${}_A B_B$ —— " —— B-mod $\to$ A-mod Restriction

For any ring ext Ind $\dashv$ Res ie $\mathrm{Hom}_B(\mathrm{Ind}\, M, N) \cong \mathrm{Hom}_A(M, \mathrm{Res}\, N)$

det. by unit + counit of adjunction $\mathrm{Hom}_B(\mathrm{Ind}\,\mathrm{Res}\, M, M) \cong \mathrm{Hom}_A(\mathrm{Res}\, M, \mathrm{Res}\, M) \ni \mathbb{1}_M$

get natl trans $\mathrm{Ind} \circ \mathrm{Res} \to \mathbb{1}_{B\text{-mod}}$

Counit ${}_B B \otimes_A B_B \to B$ just multiplication

Sim, unit is $\mathbb{1}_{A\text{-mod}} \to \mathrm{Res} \circ \mathrm{Ind}$

${}_A A_A \to {}_A B_A$ just inclusion

} satisfy some natural conditions spelled out later

For Frob ext, Res $\dashv$ Ind. Now have maps in other direction

$${}_A B_A \to {}_A A_A \qquad \partial$$

$${}_B B_B \to {}_B B \otimes_A B_B \qquad \Delta(1) = \sum b_i \otimes b_i^*$$ indep of choice of dual bases

With the grading included, Ind $\dashv$ Res $\dashv$ Ind$(2l)$, but have nice intermediate description –

consider functors Ind and Res(l)

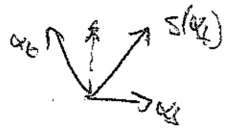
$B \otimes_A B(l) \to B$ degree $+l$

$B \to B \otimes_A B(l)$ degree $+l$

$B(l) \to A$ degree $-l$

$A \to B(l)$ degree $-l$

Let's do it in examples: $R^S \subset R$. Now $\alpha_s^2 \in R^S$.



$\alpha_t \leftrightarrow \alpha_t^\vee = 2\cos\frac{\pi}{m_{st}}\alpha_t$

so $\alpha_t + \cos\frac{\pi}{m}\alpha_s \in R^S$

Claim: $R^S = \mathbb{R}[\alpha_s^2, \{\alpha_t + \cos\frac{\pi}{m_{st}}\alpha_s\}_{t \neq s}]$

Def: $\partial_s : R \to R^S$ Demazure or divided difference operator $\deg = -2$

$\partial_s(f) = \dfrac{f - s(f)}{\alpha_s}$. Clearly in $\mathrm{f.f.}(R)^S$. Numerator in $R^{-S} = \{f \in R \mid s(f) = -f\}$

Claim: $R^{-S} = R^S \cdot \alpha_s$.

Check: ∂_s is R^S-linear Fact: $\partial_s(R^S) = 0$. Check: $\{1, \frac{\alpha_s}{2}\}$ and $\{\frac{\alpha_s}{2}, 1\}$ are dual bases

~~Check~~ Note: $\partial_s(\alpha_t) = a_{st}$. $\leftarrow$ "Cartan matrix encodes Frob. ext. structures"

More about ∂_s: ① Twisted Leibniz rule: $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$

② Braid relation $\partial_s\partial_t\cdots = \partial_t\partial_s\cdots$ so $\partial_w = \partial_{\underline{w}}$ for rex, indep of choice of rex.

③ $R \cong R^s \oplus R^{-s} \underset{R^s\text{-bim}}{\cong} R^s \oplus R^s(-2)$, $R^{-s} = R^s\alpha_s$

$f \mapsto (g,h)$ i.e. $f = g + h\frac{\alpha_s}{2}$ for $g,h \in R^s$ $\quad h = \partial_s(f)$, $\quad g = \partial_s\left(\frac{f\alpha_s}{2}\right)$

Ex 2: W Weyl gp. $R/R^W_+ \cong H^*(Fl) \cong C$ finite dim ring. Has trace map ($Fl = G/B$)

$\partial_{w_0} = \partial : C \to \mathbb{R}$ integrate against top class (general for coh. of sm. proj. vty)

Dual bases given by Schubert calculus. Top class is $\prod_{\alpha\in\Phi_+} \alpha$ (pos roots), one basis given by $\partial_w(\mathrm{Top})$.

We're interested in relative version, $R^W \subset R = H^*_B(pt)$, $R^W = H^*_G(pt)$, have $\partial_W = \partial_{w_0} : R \to R^W$

however, Schubert basis no longer gives dual bases! $\partial_{w_0}(\sigma_i\sigma_j^*) \in R^W_+$ but not nec. 0.

~~Problem~~ No nice closed formulas known!!!

Open Problem: Find Root-theoretic descriptions of dual bases for $R^I \subset R^J$, for all Cox gps

§ S. Bim

Def: $B_s = R \otimes_{R^s} R(1)$ as R-bimod. $\mathrm{Ind}\circ\mathrm{Res}(1)$ self-biadjoint.

(1) means lowest deg in degree -1.

Visualize as semiproof wall $\{ f_1 | f_2 \}$. R^s can cross through

If $f = g + h\frac{\alpha_s}{2}$ for $g,h \in R^s$ then $f| = |g + \frac{\alpha_s}{2}|h$ ($|$ means 1)

So as right R-mod, B_s has basis $\left\{ \underset{1\otimes 1}{|} \text{ and } \underset{\frac{\alpha_s}{2}\otimes 1}{\frac{\alpha_s}{2}|} \right\}$

Exercise: Why free?

Def: A Bott-Samelson bimod is $BS(\underline{w}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u = R \otimes_{R^s} R \otimes_{R^t} \cdots \otimes_{R^u} R(d)$

$\{f_1 | f_2 | \cdots | f_{d+1}\}$

Exercise: $BS(\underline{w})$ has basis $\{\alpha_{s_1}^{\varepsilon_1}\otimes\alpha_{s_2}^{\varepsilon_2}\otimes\cdots\otimes\alpha_{s_d}^{\varepsilon_d}\otimes 1\}_{\underline{\varepsilon}\in\{0,1\}^d}$ as right R-mod

Examination: (1) $B_s\otimes B_s = R\otimes_{R^s} R\otimes_{R^s} R(2) \cong R\otimes_{R^s}(R^s\oplus R^s(-2))\otimes_{R^s} R(2) = R\otimes_{R^s}R(0)\oplus R\otimes_{R^s}R(2)$ (4)

$= B_s(-1)\oplus B_s(+1)$

$f\otimes g\otimes h \longmapsto \left(f\partial_s(g)\otimes h,\ f\partial_s(g\tfrac{\alpha_s}{2})\otimes h\right)$

$\left(a\otimes\tfrac{\alpha_s}{2}\otimes b + c\otimes 1\otimes d\right) \longleftarrow (a\otimes b,\ c\otimes d)$

Categorifies $\underline{H}_s\underline{H}_s = \underline{H}_s v^{-1} + \underline{H}_s v$

(2) $B_s\otimes B_t$ can slide f_2 out of middle since R^s and R^t generate R. (unless $a_{st}=2$)

$f_1|f_2\}f_3$ So as bimod, generated by $1\otimes1\otimes1$

$\exists$ surjective map $R\otimes_{R^{st}} R(2) \longrightarrow B_s\otimes B_t$

$f\otimes g \longmapsto f\otimes1\otimes g$

Is there an inverse map?? When $m_{st}=2$ yes. $\partial_s\partial_t=\partial_t\partial_s$

Else no.

What to do with $1\otimes f\otimes 1$? unique way to slide out??

$m_{st}=2$ categorifies $\underline{H}_s\underline{H}_t=\underline{H}_t\underline{H}_s$

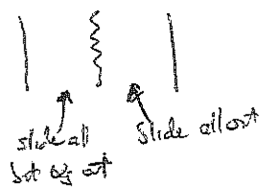
(3) $\underbrace{B_s\otimes B_t\otimes \dots}_{m_{st}}$ has $R\otimes_{R^{st}}R(m_{st})$ as a summand!

(4) $R\xrightarrow{\Delta} B_s$ unique linear elt (up to scalar) w/ $f c_s = c_s f$.

$1\longmapsto \frac{\alpha_s}{2}\otimes1+1\otimes\frac{\alpha_s}{2} = c_s$

draw as  since creates a gap for f to slide thru.

(5) $B_s\otimes B_t\otimes B_s$ $m_{st}=3$.



gen as R-bim by $1\otimes1\otimes1\otimes1 \twoheadleftarrow R\otimes_{R^{st}}R(3)$

and $\boxed{1\otimes c_t\otimes 1 \cong B_s}$

Claim: $B_s\otimes B_t\otimes B_s \cong B_s \oplus R\otimes_{R^{st}}R(3)$

Categorifies $\underline{H}_s\underline{H}_t\underline{H}_s = \underline{H}_s + \underline{H}_{sts}$

Thm (Soergel): $\mathbb{H}_W \longrightarrow [R\text{-bimod}]$ is an injective homomorphism of $\mathbb{Z}[v^{\pm1}]$-alg.

$\underline{H}_s \longmapsto [B_s]$

More categorical versions:

Def: A <u>Soergel bim</u> is a $(\oplus, \otimes, (n))$ of a <u>summand</u> of a BS Bim. Form a full add monoidal subcat of R-bim

Thm (SCT): ① $\exists!$ indec. summand $B_{\underline{w}} \overset{\oplus}{\subset} BS(\underline{w})$ which does not appear in $BS(\underline{y})$ for shorter $\underline{y}$

② $\exists$ canonical isom $B_{\underline{w}} \cong B_{\underline{w}'}$ when $\underline{w} = \underline{w}'$. So just write as B_w.

③ $\{B_w(n)\}_{n \in \mathbb{Z}}$ forms complete list of non isom indec. in SBim.

$\Rightarrow$ $[SBim] = \mathbb{Z}[v^{\pm}]\langle [B_w] \rangle$ ← basis

④ $H_W \xrightarrow{\sim} [SBim]$ is an isom. (upper triangularity)

⑤** $Hom(B, B')$ is free as right R-mod w/ graded rank $([B], [B'])$. Soergel Hom Formula.

Ex:
- $Hom(R, R) = R$ $\quad (1, 1) = 1$
- $Hom(R, B_s) = R(-1)$ $\quad (1, \underline{H}_s) = \varepsilon(H_s + v) = v$
- $Hom(B_s, B_s) = R \oplus R(-2)$ $\quad (\underline{H}_s, \underline{H}_s) = (1, \underline{H}_s^2) = \varepsilon(\underline{H}_s^2) = \varepsilon((v+v^{-1})\underline{H}_s) = v^2 + 1$.

  identity and left mult by α_s

- $Hom(B_s B_s, B_s)$ has grk $\boxed{v^{-1}} + 2v + v^3$

  ← maps of negative degree.

  proj to $B_s(1)$ and $B_s(-1)$ $\qquad$ then left mult by α_s