Some notes (KNR)

0.1. **The nil-Hecke algebra and the Yang-Baxter equation (YBE).** Let $\mathscr{H}^{n+1}$ denote the nil-Hecke algebra, that is, the algebra generated by non-commuting variables $u_1, \ldots, u_n$ (where $n+1$ is the superscript in $\mathscr{H}^{n+1}$), and subject to the following sets of relations, with respective labels (N), (C), and (B) (N stands for "nil", C for "commuting", and B for "braid"):

$$\begin{array}{lcl} \text{(N):} & u_i^2 = 0 & 1 \leq i \leq n \\ \text{(C):} & u_i u_j = u_j u_i & 1 \leq i, j \leq n, \quad |j-i| > 1 \\ \text{(B):} & u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} & 1 \leq i < n \end{array}$$

We define a map from the symmetric group $\mathfrak{S}_n$ to $\mathscr{H}^n$ as follows: given u in $\mathfrak{S}_n$, we choose a reduced expression $s_{i_1} s_{i_2} \cdots$ for u, and take the image of u to be $u_{i_1} u_{i_2} \cdots$. This map is well-defined because relations (C) and (B) hold in $\mathscr{H}^n$ and the graph of reduced expressions of any permutation is connected (the latter fact was proved by Vijay Ravikumar with the aid of wiring diagrams). We abuse notation and denote by u the image under this map of any permutation u.

We note (accept without proof) the fact that $\mathscr{H}^n$ is freely generated as a module over the coefficient ring by permutations u in $\mathfrak{S}_n$. Set $h_i(x) := 1 + u_i x$ (for arbitrary coefficient x). A routine verification shows that the following analogues of the relations above hold in $\mathscr{H}^n$:

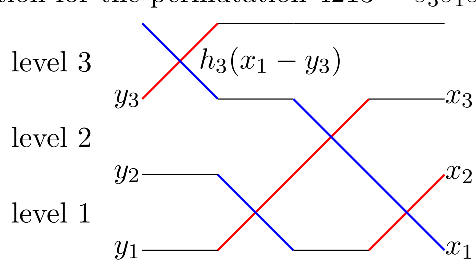
$$\begin{array}{lcl} \text{(N):} & h_i(x) h_i(y) = h_i(x+y) & 1 \leq i \leq n \\ \text{(C):} & h_i(x) h_j(y) = h_j(y) h_i(x) & 1 \leq i, j \leq n, \quad |j-i| > 1 \\ \text{(B):} & h_i(x) h_{i+1}(x+y) h_i(y) = h_{i+1}(y) h_i(x+y) h_{i+1}(x) & 1 \leq i < n \end{array}$$

This analogue (B) is called the Yang-Baxter Equation (YBE).

In the sequel we denote by $\mathscr{H}$ the algebra $\mathscr{H}^n[x, y]$, where the latter just means that the coefficient ring has been augmented by adding two sets of (commuting) variables x and y (both labeled by positive integers).

0.2. **The map ϕ from configurations to $\mathscr{H}$.** Let u be a permutation in $\mathfrak{S}_n$ and C be a wiring diagram for u. We assume tacitly that any such diagram represents a reducecd experssion for u, or, what amounts to the same, that any two wires cross at most once. Such diagrams are called "configurations" in Manivel. We associate to C an element of $\mathscr{H}$ denoted by $\phi(C)$ by a certain procedure, which we now describe by means of an example.

Shown below is a configuration for the permutation $4213 = s_3 s_1 s_2 s_1$[1]



To each crossing we associate an element $h_\ell(x_p - y_q)$ of $\mathscr{H}$ as follows: ℓ refers to the level of the crossing, the x_p refers to the "x-weight" of the Southeast strand at the crossing (coloured blue) and y_q to the "y-weight" of the Northeast strand (coloured red). We then define $\phi(C)$ to be the product of the elements of $\mathscr{H}$ associated to the crossings as we move from left to right. In the above example, $\phi(C) = h_3(x_1 - y_3) h_1(x_2 - y_1) h_2(x_1 - y_1) h_1(x_1 - y_2)$.

[1] If u is the underlying permutation of a configuration, then $u(i)$ is obtained as follows: at the right end, choose the i^{th} strand from below; follow it to the left end of the configuration; determine which strand it is, counting from below; suppose it is the j^{th} strand; then $u(i) = j$.

Thanks to the relations ((N), (C), and YBE) in $\mathscr{H}^n$ and the connectedness of the graph of reduced expressions of any permutation, we conclude:

Theorem 1. *The association ϕ depends only upon the underlying permutation u and not on the specific configuration C.*

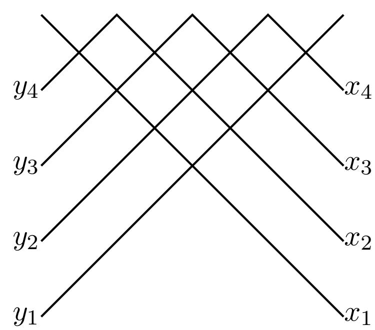
0.3. **Double Schubert polynomials.** Set $\Delta = \Delta^n(x,y) = \prod_{1\leq i,j\leq n,\ i+j\leq n}(x_i - y_j)$, where the x and y are the same (commuting) variables we added to the coefficient ring of $\mathscr{H}$ above. We define the *double Schubert polynomial* associated to a permutation u of $\mathfrak{S}_n$ by:

(1) $$S_u = S_u(x,y) := \partial_{u^{-1}w_0}\Delta \qquad \text{(the operators } \partial \text{ act only on the } x \text{ variables)}$$

This makes sense for the same familiar reason: the divided difference operators ∂_i satisfy the commuting and braid relations, and the graph of reduced expressions for u is connected.

By definition, these (double) Schubert polynomials are elements of the coefficient ring of $\mathscr{H}$. Since the operators ∂_i also satisfy (N), it follows that we get an algebra homomorphism from $\mathscr{H}$ to a (suitably defined) ring of operators on the coefficient ring by mapping u_i to ∂_i.

0.4. **The Schubert sweater $C_{\mathbf{Sch}}$ and coefficients of $\phi(C_{\mathbf{Sch}})$.** We now arrive at what seems to be by far the most important technical result of Chapter 2 in Manivel (namely his 2.3.7). The SCHUBERT SWEATER C_{Sch} is the following very particular configuration for the longest permutation w_0 in $\mathfrak{S}_n$ (drawing shows the case $n = 5$):



Let us look at the expression for $\phi(C_{\text{Sch}})$ as a linear combination of the basis elements u, $u \in \mathfrak{S}_n$, of $\mathscr{H}$. It is easy to see that the coefficient of w_0 is Δ, which of course is the double Schubert polynomial $S_{w_0}(x,y)$. What about the other coefficients? Every coefficient is precisely the double Schubert polynomial:

[t:sweatschub]

Theorem 2.

$$\boxed{\phi(C_{\text{Sch}}) = \sum_{u\in\mathfrak{S}_n} S_u(x,y)u}$$

The proof involves manipulations with relations in $\mathscr{H}$ which we skip for the moment and proceed, pausing only to record the following corollary (of the proof).

[c:expr]

Corollary 3.

$$\boxed{\phi(C_{\text{Sch}}) = S(y)^{-1}S(x)}$$

where

$$S(x) := H_1(x_1)H_2(x_2)\cdots H_{n-1}(x_{n-1}) \text{ and } H_i(x) = h_{n-1}(x)h_{n-2}(x)\cdots h_i(x)$$

Putting the theorem and corollary together and setting $y = 0$, we obtain:

(2) $$S(x) = \sum_{u\in\mathfrak{S}_n} S_u(x)u$$

where $S_u(x)$ is the simple Schubert polynomial. From this we conclude:

Corollary 4. *The Fomin-Kirillov recipe for computing the simple Schubert polynomials is justified.*[2]

In turn we conclude that Schubert polynomials have non-negative coefficients. In fact, we get a proof of Stanley's conjecture (for the coefficients of a Schubert polynomial).

Exercise 5. Show that $S(y)^{-1} = \sum_{u\in\mathfrak{S}_n} \epsilon(u) S_{u^{-1}}(y) u = \sum_{u\in\mathfrak{S}_n} S_{u^{-1}}(-y) u$.

Let us look at Corollary 3 and equate the coefficient of w_0 on both sides of the equality. By Theorem 2, this coefficient on the left side is Δ. Using the expressions for $S(x)$ and $S(y)^{-1}$ above to compute the coefficient on the right, we get:

Corollary 6. (Cauchy identity)

$$\prod_{i+j\leq n} (x_i - y_j) = \sum_{v\in\mathfrak{S}_n} S_v(x) S_{vw_0}(-y)$$

We will derive a consequence from this to be applied in the next theorem. Let us permute the x variables by a permutation u, which means that we replace x_i by $x_{u(i)}$. Let us permute the y variables too, by w_0. Let us then put $y = x$. Then the left hand side is zero except when u is the identity, in which case it is the van der Monde determinant. We thus have:

$$\text{van der Monde determinant} \times \delta_{u,\mathrm{id}} = \sum_{v\in\mathfrak{S}_n} \epsilon(v)\epsilon(w_0)\ uS_v(x)\ w_0 S_{vw_0}(x) \tag{3}$$

Theorem 7. *The coinvariant ring is the regular representation of $\mathfrak{S}_n$.*

Proof: Let us compute the character (trace) of (the action of) a permutation u in the basis S_w. The coefficient in the row corresponding to w and column corresponding to v is $\langle uS_v, S_w^*\rangle$. The trace is therefore given by $\sum_{v\in\mathfrak{S}_n} \langle uS_v, S_v^*\rangle$, which we want to show is $\delta_{u,\mathrm{id}} n!$. When $u = \mathrm{id}$, it is of course clear that the character equals the dimension of coinvariant ring (namely, $n!$). To show that it is zero when $u \neq \mathrm{id}$, we plug in the value of S_v^* as in the next exercise and use Equation (3):

$$\sum_{v\in\mathfrak{S}_n} \langle uS_v, S_v^*\rangle = \partial_{w_0}\left(\sum_{v\in\mathfrak{S}_n} \epsilon(v)\epsilon(w_0)\ uS_v(x)\ w_0 S_{vw_0}(x)\right)(0) = 0 \quad (\text{when } u \neq \mathrm{id})$$

□

Exercise 8. Verify the following:

(1) $w_0 \partial_i w_0 = -\partial_{n-i}$ and more generally $w_0 \partial_u w_0 = \epsilon(v) \partial_{w_0 v w_0}$ (the first is obtained from the second by putting $u = s_i$).

(2) the dual basis (under the form we defined) of the basis of $\{S_v\}$ of Schubert polynomials for the coinvariant ring is given by: $\boxed{S_v^* = \epsilon(v)\epsilon(w_0)\ w_0 S_{vw_0}}$

Solution:

(1) Observe that $w_0 s_i w_0 = s_{n-i}$ and $w_0 x_i = x_{n+1-i}$. We thus have

$$w_0 \partial_i w_0 = w_0 \frac{w_0 - s_i \circ w_0}{x_i - x_{i+1}} = \frac{1 - w_0 s_i w_0}{w_0(x_i - x_{i+1})} = -\frac{1 - s_{n-i}}{(x_{n-i} - x_{n-i-1})}$$

[2]We do in fact have a Fomin-Kirillov recipe for computing the double Schubert polynomials.

(2) It is clearly enough to show that $\epsilon(v)\epsilon(w_0)\langle S_u, w_0 S_{vw_0}\rangle = \delta_{u,v}$.

$$\begin{aligned}
\text{left hand side} &= \epsilon(v)\epsilon(w_0)\langle \partial_{u^{-1}w_0} x^\delta, w_0 S_{vw_0}\rangle && \text{(by definition of } \partial_u\text{)} \\
&= \epsilon(v)\epsilon(w_0)\langle x^\delta, \partial_{w_0 u} w_0 S_{vw_0}\rangle && (\langle \partial_u P, Q\rangle = \langle P, \partial_{u^{-1}} Q\rangle) \\
&= \epsilon(v)\epsilon(w_0)\langle x^\delta, w_0\ w_0 \partial_{w_0 u} w_0 S_{vw_0}\rangle && \text{since } w_0 w_0 = \text{id} \\
&= \epsilon(v)\epsilon(u)\langle x^\delta, w_0 \partial_{uw_0} S_{vw_0}\rangle && \text{(since } w_0 \partial_{w_0 u} w_0 = \epsilon(u)\epsilon(w_0)\partial_{uw_0}\text{)} \\
&= \epsilon(v)\epsilon(u)\partial_{w_0}\left(x^\delta \cdot w_0 \partial_{uw_0} S_{vw_0}\right)(0) && \text{(by definition of the form } \langle\ ,\ \rangle\text{)}
\end{aligned}$$

Note that $\partial_{uw_0} S_{vw_0}$ equals 0 if $\ell(uw_0) \geq \ell(vw_0)$ except if $u = v$ (in which case it is 1). If $\ell(uw_0) < \ell(vw_0)$ then $\partial_{uw_0} S_{vw_0}$ is homogeneous of positive degree (possibly zero), so $\partial_{w_0}(x^\delta \cdot w_0 \partial_{uw_0} S_{vw_0})$ is homogeneous of positive degree, so it has zero constant term. This finishes the proof that $\langle S_u, S_v^*\rangle = \delta_{u,v}$ for S_v^* as in the claim. $\square$