Partitions

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svis @ imsc. res. in]

(a) Examples

$n=1$	$n=2$	$n=3$	$n=4$
1	2	3	4
	1+1	2+1	3+1
		1+1+1	2+2
			2+1+1
			1+1+1+1

$p(n)$ = number of partitions of n ("the partition function")

n	$p(n)$
1	1
2	2
3	3
4	5
5	7
6	11
⋮	⋮

Grows very fast [See: http://oeis.org/A000041]

Some History: Euler – 1700's (generating function), identities

Sylvester, Rogers, MacMahon – late 1800s / early 1900s

Ramanujan – Hardy – 1914 and Rademacher – 1940: EXACT formula for $p(n)$.

Bruinier – Ono – 2011

8 Special partitions

Type A Partitions in which every part is ≤ 3

(eg) $n=6$:

$3+3$
$3+2+1$
$3+1+1+1$
$2+2+2$
$2+2+1+1$
$2+1+1+1+1$
$1+1+1+1+1+1$

total number $=7$

Type B Partitions in which number of parts is ≤ 3

(eg) $n=6$:

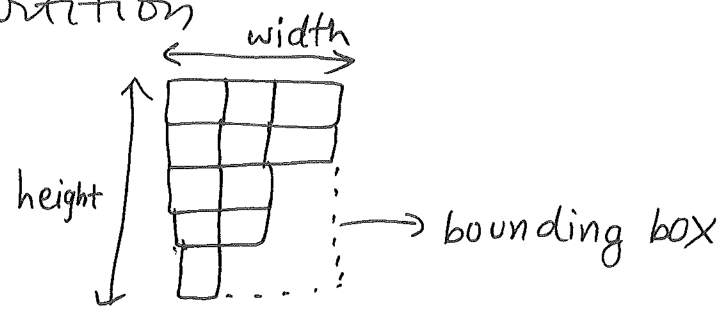
6
$5+1$
$4+2$
$4+1+1$
$3+3$
$3+2+1$
$2+2+2$

total $=7$

Theorem: Number of Type A partitions of n $=$ Number of Type B partitions of n for every natural number n.

$\boxed{8}$ Diagram of a partition

(eg) $3+3+2+2+1$



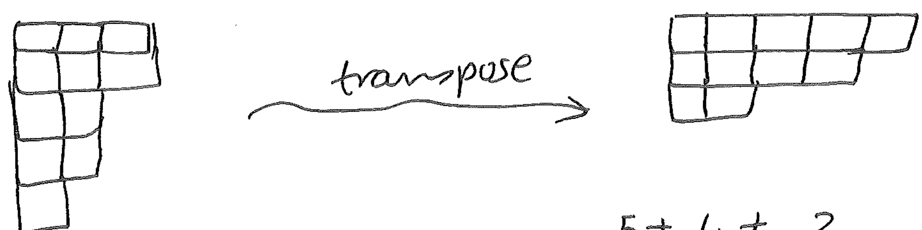
Type A partition $\iff$ diagram has width ≤ 3.
Type B partition $\iff$ diagram has height ≤ 3.

<u>Question</u>: Is there an operation that will convert a type A diagram into a type B diagram (and vice-versa)?

<u>Answer</u>: Yes; <u>TRANSPOSE</u> (just like for matrices)

<u>i.e.</u>, Read the partition along the columns rather than along the rows.

(eg)



$3+3+2+2+1$ $\qquad\qquad$ $5+4+2$

<u>Type A</u>	$\xrightarrow{\text{transpose}}$	<u>Type B</u>	
$3+3$	$\longrightarrow$	$2+2+2$	
$3+2+1$	$\longrightarrow$	$3+2+1$	(own transpose).
$3+1+1+1$	$\longrightarrow$	$4+1+1$	
$2+2+2$	$\longrightarrow$	$3+3$	
$2+2+1+1$	$\longrightarrow$	$4+2$	
$2+1+1+1+1$	$\longrightarrow$	$5+1$	
$1+1+1+1+1+1$	$\longrightarrow$	6	

"BIJECTIVE PROOF" of our theorem.

Type C: Difference between consec. parts is ≥ 2

(eg) $n = 9$

9

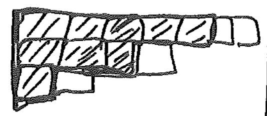
8+1

7+2

6+3

5+3+1

Diagram looks like (has an inscribed zig-zag shape)



Type D: Every part is $\geq$ no. of parts.

(eg) 9

~~8+1~~

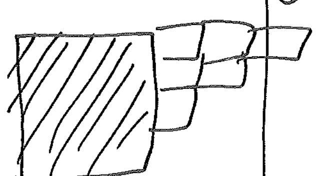
7+2

6+3

5+4

3+3+3

Diagram



has an inscribed square

Thm: no. of type C partns = no of type D partns $\forall n$.

Pf: operation that converts type C dig to type D.

(Q) How many boxes in zig-zag shape of ht d

Ans: $1+3+5+\ldots+(2d-1)$

$= d^2$

$= \#$ boxes in a $d \times d$ square

∴ operation

1. Take a type C dig
2. cut out the zig-zag
3. Rearrange into square
4. Paste remaining boxes in the same way.

Done!

Table

type C	opern	type D
9	$\longrightarrow$	9
8+1	$\longrightarrow$	7+2
7+2	$\longrightarrow$	6+3
6+3	$\longrightarrow$	5+4
5+3+1	$\longrightarrow$	3+3+3.

Type E: All parts ending in 1 ~~or~~ 4, 6, 9. (eg) $\left\{\begin{array}{l} 9 \\ 6+1 \\ 4+4+1 \\ 4+1 \\ 1 \end{array}\right.$ (RR ids)

Further reading

① Andrews - Eriksson — Integer partns

(~~make [illegible] copies~~)

Theorem: (Rogers - Ramanujan)

number of Type C partitions = no. of Type D partitions = no. of Type E partitions.