A note on the grand theorems of functional analysis

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Abstract

Three grand theorems of functional analysis are the uniform boundedness (or, Banach-Steinhaus) theorem, the open mapping theorem and the closed graph theorem. All these are consequences of a topological result known as Baire's (category) theorem. The mathematical literature has several results which show the interconnections between these theorems, some well known and well documented, while some others are less known. The aim of this note is to present all these interdependences in a cogent manner.

1 Introduction

The high point in any first course on functional analysis is the proof of three grand theorems, known as the uniform boundedness theorem, the open mapping theorem and the closed graph theorem. All three depend on the completeness of some or all of the spaces involved and their proofs depend on a topological result known as Baire's thoerem (or, the Baire category theorem). It is well recorded that the open mapping theorem and the closed graph theorem are equivalent in the sense that each can be deduced from the other, though in most text books one of them is proved, starting from Baire's theorem, and the other is deduced as a consequence. The uniform boundedness theorem is proved independently. Looking at the mathematical literature, we also see works wherein it is shown how to deduce the uniform boundedness theorem from the closed graph theorem. The converse is also true with some restrictions. These results seem to be less known. The aim of this note is to present all the possible equivalences and deductions in a cogent manner. We will also show that all the three results are 'equivalent' to each other in the case of Hilbert spaces.

It needs to be stressed again that all these results are scattered in the literature and no claim to originality in the proof techniques is made.

The proof of the uniform boundedness theorem from Baire's theorem is probably the simplest of all these proofs. However, there also exist several 'elementary' proofs of this result, in the sense that these proofs do not use Baire's theorem. We will present here a really simple proof, which does not involve Baire's theorem, of the uniform boundedness theorem. We do not present any direct proofs of the open mapping or closed graph theorems as these could be found in any text book on functional analysis.

2 The main result

Theorem 2.1 *Each of the following statements imply the others.*

(i) **The closed graph theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a linear map. If the graph of T defined by

$$G(T) \ = \ \{(x, Tx) \mid x \in V\} \ \subset \ V \times W$$

is closed in $V \times W$, then T is continuous.

(ii) **The open mapping theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The bounded inverse theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) The two norms theorem
Let V be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. If V is a Banach space with respect to either norm and if there exists a constant $C > 0$ such that

$$\|x\|_1 \;\leq\; C\|x\|_2 \tag{2.1}$$

for every $x \in V$, then the two norms are equivalent.

Each of the above statements implies the following:

(v) **Uniform boundedness theorem**
Let V be a Banach space and W be a normed linear space. Let, $T_i : V \to W$ be a continuous linear map for each $i \in I$. If

$$\sup_{i \in I} \|T_i x\|_W \;<\; \infty \tag{2.2}$$

for each $x \in V$, then there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for each $i \in I$.

Proof: (i) $\Rightarrow$ (ii)
Assume that the closed graph theorem is true. Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear map which is surjective. Let $N = \mathrm{Ker}(T)$, which is a closed subspace. Let $y \in W$. Since T is onto, there always exists $x \in V$ such that $Tx = y$. Define $Sy = x + N \in V/N$. If $Tx_1 = Tx_2 = y$, then $x_1 - x_2 \in N$ and so $x_1 + N = x_2 + N$ and so the map S is well-defined, and clearly linear. We claim that the graph of S is closed. To see this, let $y_n \to y$ in W and let $Sy_n \to z + N$ in V/N. Let $Tx_n = y_n$ so

that $Sy_n = x_n + N$. Then it follows that

$$\inf_{w \in N} \|x_n - z - w\|_V \ \to \ 0$$

as $n \to \infty$. Thus, there exists a sequence $\{w_n\}$ in N such that $\|x_n - z - w_n\|_V \to 0$, or in other words, $x_n - w_n \to z$. Then $y_n = T(x_n) = T(x_n - w_n) \to Tz$. Thus $y = Tz$. Then it follows that $Sy = z + N$, which establishes our claim.

Consequently, by the closed graph theorem, S is continuous, *i.e.* there exists $K > 0$ such that for all $y \in W$, we have $\|Sy\|_{V/N} \leq K\|y\|_W$. Thus $\|y\|_W < \frac{1}{K}$ implies that $\|Sy\|_{V/N} < 1$, or, in other words, if $\|y\|_W < \frac{1}{K}$, for $y \in W$, there exists $x \in V$ and $z \in N$ such that $\|x - z\|_V < 1$ and $y = Tx = T(x - z)$. Thus it follows that

$$B_W\left(0; \frac{1}{K}\right) \subset T(B_V(0;1)$$

(where $B_X(x;r)$ denotes the open ball centered at a point x and of radius $r > 0$ in a normed linear space X). From this we can easily conclude that T maps open sets of V onto open sets of W.

(ii) $\Rightarrow$ (iii)
Assume that the open mapping theorem is true. Then if $T : V \to W$ is a bijective and continuous linear map between the Banach spaces V and W, then it maps open sets onto open sets. This immediately implies that $T^{-1} : W \to V$ is continuous since, if $U \subset V$ is an open set, then $(T^{-1})^{-1}(U) = T(U)$ which is open in W.

(iii) $\Rightarrow$ (iv)
Let V be a vector space which is complete with respect to both the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If $I : V \to V$ is the identity mapping, then the given inequality (2.1) implies that this map is a continuous bijection from the Banach space V (with the norm $\|\cdot\|_2$) to the Banach space V (with the norm $\|\cdot\|_1$) and hence, by (iii), it is an isomorphism, which establishes the equivalence of the two norms.

(iv) $\Rightarrow$ (i)
Let V and W be Banach spaces and let $T : V \to W$ be a linear map whose

graph is closed. Define the norm

$$\|x\|_1 \;=\; \|x\|_V + \|Tx\|_W$$

for $x \in V$. If $\{x_n\}$ is a Cauchy sequence in V with respect to this norm, then it is also a Cauchy sequence with respect to the orignal norm $\|\cdot\|_V$. Further $\{Tx_n\}$ is a Cauchy sequence in W. Since the spaces are complete with the original norms, it follows that $x_n \to x$ in V and $Tx_n \to y$ in W. Since the graph is closed this implies that $Tx = y$. Thus $x_n \to x$ in V with respect to the norm $\|\cdot\|_1$. Thus we deduce that V is a Banach space with respect to the norm $\|\cdot\|_1$ as well. Clearly, for each $x \in V$, we have

$$\|x\|_V \;\leq\; \|x\|_1.$$

Thus the two norms are equivalent by virtue of (iv) and so there exists a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_V$ for each $x \in V$. In particular, we also have $\|Tx\|_W \leq C\|x\|_V$, which proves the continuity of T.

We now deduce (v) assuming the validity of the preceding statements. Let $\{T_i\}_{i\in I}$ be a collection of continuous linear maps from a Banach space V to a normed linear space W such that (2.2) holds for all $x \in V$. Now consider the norm on V defined by

$$\|x\|_2 \;=\; \|x\|_V + \sup_{i\in I}\|T_i x\|_W,\;\; x \in V$$

which is well-defined in view of (2.2). If $\{x_n\}$ is a Cauchy sequence in V with respect to this norm, then it is also a Cauchy sequence with the original norm and so, $x_n \to x$ in V (with respect to the original norm). From the definition of $\|\cdot\|_2$, it follows that the sequences $\{T_i x_n\}$ are uniformly Cauchy from which we immediately see that

$$\sup_{i\in I}\|T_i x_n - T_i x\|_W \;\stackrel{n\to\infty}{\longrightarrow}\; 0.$$

Thus $\{x_n\}$ converges to x with respect to the norm $\|\cdot\|_2$ as well and so the space V is complete with respect to both norms. Since we have that $\|x\|_V \leq \|x\|_2$ for all $x \in V$, it follows from (iv) that these norms are equivalent. Thus there exists $C > 0$ such that for all $x \in V$, we have $\|x\|_2 \leq C\|x\|_V$. In particular, we have

$$\sup_{i\in I}\|T_i x\|_W \;\leq\; C\|x\|_V$$

for all $x \in V$ which implies that $\|T_i\| \leq C$ for every $i \in I$. This completes the proof. ■

Remark 2.1 In many books on functional analysis, the open mapping theorem is proved using Baire's theorem and the closed graph theorem is deduced in the same way we have proved the implications (ii)⇒(iii)⇒(iv)⇒(i). The uniform boundedness theorem is proved independently, and in a fairly simple manner, from Baire's theorem. The proof of the implication (iv)⇒(v) appears in a note by Ramaswamy and Ramasamy [2]. ■

The proof of the uniform boundedness theorem starting from Baire's theorem is, as already remarked, fairly simple and appears in all texts on functional analysis. We now present a very simple proof (due to Sokal [4]) of this theorem without appealing to Baire's theorem.

Proof of the uniform boundedness theorem:

Step 1. Let $x, y \in V$. Since $2y = (x+y) - (x-y)$, the triangle inequality gives

$$\|Ty\|_W \leq \frac{1}{2}[\|T(x+y)\|_W + \|T(x-y)\|_W] \leq \max\{\|T(x+y)\|_W, \|T(x-y)\|_W\}$$

where T is a continuous linear map from V to W. If we now take the supremum as y varies over the open ball centred at the origin and of radius $r > 0$, we deduce that

$$\sup_{x' \in B_V(x;r)} \|Tx'\|_W \geq r\|T\|. \tag{2.3}$$

Step 2. Assume that $\{\|T_i\|\}_{i \in I}$ is unbounded. Then choose a sequence $\{T_n\}_{n=1}^\infty$ from this family such that $\|T_n\| \geq 4^n$ for each n. Set $x_0 = 0$. Then, by Step 1, we can inductively find $\{x_n\}_{n=1}^\infty$ in V such that $\|x_n - x_{n-1}\|_V < 3^{-n}$ and $\|T_n x_n\|_W > \frac{2}{3}3^{-n}\|T_n\|$. By construction, since $\sum 3^{-n}$ is convergent, the sequence $\{x_n\}$ is Cauchy and since V is complete, we have $x_n \to x$ in V.

Now, if $m > n$, we have

$$\|x_n - x_m\|_V \leq 3^{-(n+1)} + 3^{-(n+2)} + \cdots + 3^{-m}.$$

Keeping n fixed and letting $m \to \infty$, we deduce that

$$\|x_n - x\|_V \leq \frac{1}{2}3^{-n}.$$

Then, by the triangle inequality, we get that

$$\|T_n x\|_W \;\geq\; \|T_n x_n\|_W - \|T_n(x_n - x)\|_W \;\geq\; \frac{1}{6}3^{-n}\|T_n\| \;\geq\; \frac{1}{6}\left(\frac{4}{3}\right)^n .$$

Thus $\{\|T_n x\|_W\}$ is unbounded which contradicts (2.2). This completes the proof. ■

3 Operators on Hilbert spaces

All the statements (i)-(v) of Theorem 2.1 are equivalent to each other in the context of Hilbert spaces. More precisely, we have the following result.

Theorem 3.1 *Each of the following statements imply the others.*

(i) **The closed graph theorem**
Let V and W be Hilbert spaces and let $T : V \to W$ be a linear map. If the graph of T is closed in $V \times W$, then T is continuous.

(ii) **The open mapping theorem**
Let V and W be Hilbert spaces and let $T : V \to W$ be a continuous linear map which is surjective. Then T is an open map.

(iii) **The bounded inverse theorem**
Let V and W be Hilbert spaces and let $T : V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) The two norms theorem
Let V be a vector space and let $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be two inner-products on V yielding the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V, respetively. If V is a Hilbert space with respect to either inner-product and if there exists a constant $C > 0$ such that

$$\|x\|_1 \;\leq\; C\|x\|_2 \tag{3.1}$$

for every $x \in V$, then the two norms are equivalent.

(v) **Uniform boundedness theorem**
Let V and W be Hilbert spaces. Let, $T_i : V \to W$ be a continuous linear map

for each $i \in I$. If

$$\sup_{i\in I} \|T_i x\|_W \ < \ \infty \tag{3.2}$$

for each $x \in V$, then there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for each $i \in I$.

Proof: The equivalence of statements (i)-(iv) has already been established in Theorem 2.1. The implication (iv)⇒(v) has also been proved there. So we just need to show that the statement (v) implies the others. We will show that (v)⇒(i). Let $T : V \to W$ be a linear map. Then we can define its adjoint $T^* : D(T^*) \subset W \to V$ as follows. We define

$$D(T^*) = \{y \in W \mid |(Tx, y)_W| \leq C\|x\|_V \text{ for all } x \in V\}.$$

If $y \in D(T^*)$, then $x \mapsto (Tx, y)_W$ defines a continuous linear functional on V and so, by the Riesz representation theorem, there exists a unique element, that we denote as T^*y, in V such that

$$(x, T^*y)_V \ = \ (Tx, y)_W. \tag{3.3}$$

The map $T^* : D(T^*) \subset W \to V$ is clearly linear. We claim that if the graph of T is closed, then $D(T^*) = W$. We show this in two steps.

Step 1: $D(T^*)$ is dense in W. If not, by the Hahn-Banach theorem, there exists $y \in W$ such that $y \neq 0$ and $(z, y)_W = 0$ for all $z \in D(T^*)$. Now $(0, y)$ does not belong to the graph of T and since the graph of T is closed, there exists $(v, w) \in V \times W$ such that $(y, w)_W \neq 0$ and $(x, v)_V + (Tx, w)_W = 0$ for all $x \in V$, again by the Hahn-Banach theorem. Thus, for all $x \in V$,

$$|(Tx, w)_W| \ \leq \ \|x\|_V\|v\|_V$$

and so $w \in D(T^*)$ which then implies that $(w, y)_W = 0$, which is a contradiction. This establishes the density of $D(T^*)$ in W.

Step 2: $D(T^*) = W$. Let $y \in W$. Then, there exists a sequence $\{y_n\}$ in $D(T^*)$ such that $y_n \to y$. Thus, for all $x \in V$,

$$(x, T^*y_n)_V \ = \ (Tx, y_n)_W. \tag{3.4}$$

Since the sequence $\{y_n\}$ is convergent, it is bounded, and we deduce from (3.4) that

$$|(x, T^*y_n)_V| \ \leq \ C\|Tx\|_W \text{ for all } x \in V.$$

It then follows from the uniform boundedness theorem that $\{T^*y_n\}$ is bounded as well. Let $\|T^*y_n\|_V \leq K$ for all n. Thus (3.4) yields

$$|(Tx, y_n)_W| \;\leq\; K\|x\|_V \text{ for all } x \in V$$

and passing to the limit, we get

$$|(Tx, y)_W| \;\leq\; K\|x\|_V \text{ for all } x \in V.$$

Thus we deduce that $y \in D(T^*)$ which establishes our claim since y was arbitrarily chosen in W.

Thus, from (3.3), it follows that for all $x \in V$ such that $\|x\| \leq 1$, and for all $y \in W$, we have

$$|(Tx, y)_W| \;\leq\; \|T^*y\|_V.$$

Consequently, it follows, once again from the uniform boundedness theorem, that

$$\sup_{\|x\|_V \leq 1} \|Tx\|_W \;<\; \infty.$$

In other words, T is continuous. This completes the proof. ■

Remark 3.1 Similar arguments appear in the literature. Ramaswamy and Ramasamy [2] show that the uniform boundedness theorem implies the closed graph theorem provided we assume that the target space W is reflexive, while it is enough to assume that V is Banach. Ramm [3] also proves this implication when $V = W$ is a Hilbert space.

The main point to note here is that for the implication (iv)⇒(v), it was enough that V be complete while nothing need be assumed on W. For the reverse implication, we need the reflexivity (and hence the completeness) of W. The validity of the implication (v)⇒(iv), in the case of an arbitrary normed linear space W, seems to be still open. ■

Remark 3.2 In view of the simple and 'elementary' proof of the uniform boundedness theorem given in Section 2, we now have a complete proof of all the statements in Theorem 3.1 without appealing to Baire's theorem, in the framework of Hilbert spaces. ■

Remark 3.3 Halmos [1] also gives a proof of the uniform boundedness theorem in the framework of Hilbert spaces without appealing to Baire's theorem. ■

References

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