The grand theorems of functional analysis revisited: a Baire-free approach

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Abstract

The equivalence of the uniform boundedness principle, the open mapping theorem and the closed graph theorem is established. These theorems are proved without the use of the Baire category theorem.

1 Introduction

In an earlier article (cf. Kesavan [2], the following result was proved.

Theorem 1.1 *Each of the following statements imply the others.*

(i) **The closed graph theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a linear map. If the graph of T defined by

$$G(T) \;=\; \{(x, Tx) \mid x \in V\} \;\subset\; V \times W$$

is closed in $V \times W$, then T is continuous.

(ii) **The open mapping theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The bounded inverse theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) The two norms theorem
Let V be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. If V

is a Banach space with respect to either norm and if there exists a constant $C > 0$ such that

$$\|x\|_1 \;\leq\; C\|x\|_2 \tag{1.1}$$

for every $x \in V$, then the two norms are equivalent.

Each of the above statements implies the following:

(v) **Uniform boundedness theorem**
Let V be a Banach space and W be a normed linear space. Let, $T_i : V \to W$ be a continuous linear map for each $i \in I$. If

$$\sup_{i\in I} \|T_i x\|_W \;<\; \infty \tag{1.2}$$

for each $x \in V$, then there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for each $i \in I$. ■

The aim of the present note is two-fold: first we show that, in fact, all the statements (i)-(v) in the above theorem are equivalent. Secondly, since we gave a proof of the uniform boundedness principle without using the Baire category theorem in the above mentioned article, we can now prove all the theorems without the use of Baire's theorem.

It is important, at this point, to set the record straight about our motives for writing this article. First of all, no originality is claimed regarding the mathematical arguments. All the proofs presented are already available, some in standard textbooks and some scattered in the literature. The aim of writing this article is, on one hand, to present all the arguments in a cogent form in one place for ready reference, and on the other hand, to show the logical dependence of these results on each other.

It must be stressed that the aim of this article is certainly not to downplay the importance of the Baire category theorem. Indeed, it can be seen that the proofs of these results, found in standard text books, using the Baire category theorem, are easier and more natural. However, the fact that they can be proved without the use of Baire's theorem shows that it is the completeness of the spaces involved which is the basis for these theorems. Baire's theorem, while being a convenient and unifying tool to prove these results, is not the foundation for these results, as often students are led to believe.

For sake of completeness of the exposition, and for all proofs to be in a single place for ready reference, we will be presenting all the proofs. Many of them are well known and can be found in most text books.

2 The main result

Theorem 2.1 *Each of the following statements implies the others.*

(i) **Uniform boundedness theorem**
Let V be a Banach space and W be a normed linear space. Let, $T_i : V \to W$ be a continuous linear map for each $i \in I$. If

$$\sup_{i \in I} \|T_i x\|_W \ < \ \infty \tag{2.1}$$

for each $x \in V$, then there exists a constant $C > 0$ such that $\|T_i\| \leq C$ for each $i \in I$.

(ii) **The open mapping theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The bounded inverse theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) **The closed graph theorem**
Let V and W be Banach spaces and let $T : V \to W$ be a linear map. If the graph of T defined by

$$G(T) \ = \ \{(x, Tx) \mid x \in V\} \ \subset \ V \times W$$

is closed in $V \times W$, then T is continuous.

(v) The two norms theorem
Let V be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. If V

is a Banach space with respect to either norm and if there exists a constant $C > 0$ *such that*

$$\|x\|_1 \ \leq \ C\|x\|_2 \tag{2.2}$$

for every $x \in V$*, then the two norms are equivalent.*

Proof that (i) $\Rightarrow$ (ii).

Let V and W be Banach spaces and let $T : V \to W$ be a continuous linear mapping which is surjective.

Step 1. Let B denote the open unit ball in V. We show that there exists a constant $c > 0$ such that the open ball in W with centre at the origin and radius $2c$, denoted $B_W(0; 2c)$ is contained in $\overline{T(B)}$, the closure, in W, of the image of B.

For each positive integer n, define a norm on W by

$$\|y\|_n \ \stackrel{\text{def}}{=} \ \inf\{\|v\|_V + n\|w\|_W \mid v \in V, w \in W, w + Tv = y\}. \tag{2.3}$$

Since T is onto, given any y and w in W, we can always find $v \in V$ such that $w + Tv = y$. Thus the quantity $\|y\|_n$ is well-defined and it is easy to verify that this defines a norm on W. Since we can take $w = y$ and $v = 0$, so that $w + Tv = y$, we imediately see that

$$\|y\|_n \ \leq \ n\|y\|_W. \tag{2.4}$$

Further, since T is onto, given $y \in W$, there exsts $x \in V$ such that $Tx = y$ from which it follows that

$$\|y\|_n \ \leq \ \|x\|_V. \tag{2.5}$$

Thus, for each $y \in W$, the sequence $\{\|y\|_n\}$ is bounded.

Now let $\mathbb{N}$ stand for the natural numbers (*i.e.* positive integers greater than, or equal to unity). Let

$$Z \ = \ \{f : \mathbb{N} \to W \ | f \text{ is finitely supported}\}.$$

In other words, Z is the space of sequences with entries taken from W such that all but a finite number of the entries are zero. Then, it is immediate to see that the following quantity is well-defined and that it defines a norm on Z:

$$\|f\|_Z \stackrel{\text{def}}{=} \sup_n \|f(n)\|_n.$$

Define $S_n : W \to Z$ by

$$S_n(y)(k) \;=\; \begin{cases} y, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Then, using (2.4), we get

$$\|S_n(y)\|_Z \;=\; \|y\|_n \;\leq\; n\|y\|_W.$$

Thus,for each n, we have that S_n defines a continuous linear map from W (which is complete with respect to the norm $\|\cdot\|_W$) into the normed linear space Z. Further, by virtue of (2.5) we also have that, if $y = Tx$, then,

$$\|S_n(y)\|_Z \;\leq\; \|x\|_V.$$

The sequence $\{S_n\}$ is pointwise bounded and hence, by (i), it is uniformly bounded. Thus, there exists $M > 0$ such that $\|S_n\| \leq M$.

Let $c > 0$ such that $2c < M^{-1}$. Let $y \in W$ be such that $\|y\|_W < 2c$. Then,

$$\|y\|_n \;=\; \|S_n(y)\|_Z \;\leq\; \|S_n\|\;\|y\|_W \;\leq M\|y\|_W \;<\; 1.$$

Then, there exist $v_n \in V$ and $w_n \in W$ such that $w_n + Tv_n = y$ and $\|v_n\|_V + n\|w_n\|_W < 1$.This shows that $\|v_n\|_V < 1$ for all n and that $w_n \to 0$ in W. Thus $Tv_n \to y$ and so $y \in \overline{T(B)}$. Thus

$$B_W(0; 2c) \;\subset\; \overline{T(B)}. \tag{2.6}$$

Step 2. Let $c > 0$ be such that (2.6) is true. Then we claim that

$$B_W(0; c) \;\subset\; T(B). \tag{2.7}$$

Let $y \in B_W(0; c)$. We need to find $x \in B$ such that $T(x) = y$. Let $\varepsilon > 0$. There exists $z \in V$ such that $\|z\|_V < 1/2$ and $\|y - T(z)\|_W < \varepsilon$ by virtue of

(2.6) (applied to $2y$). Set $\varepsilon = c/2$ and let $z_1 \in V$ be such that $\|z_1\|_V < 1/2$ and $\|T(z_1) - y\|_W < c/2$.

We can iterate this procedure. By another application of (2.6) (to $4(T(z_1) - y)$) we can find $z_2 \in V$ such that

$$\|z_2\|_V \ < \ \frac{1}{4}, \ \|T(z_1 + z_2) - y\|_W \ < \ \frac{c}{4}.$$

Thus, we can find, by repeated use of (2.6), a sequence $\{z_n\}$ in V such that

$$\|z_n\|_V \ < \ \frac{1}{2^n}, \ \|T(z_1 + \cdots + z_n) - y\|_W \ < \ \frac{c}{2^n}.$$

Then, it follows that the sequence $\{z_1 + \cdots + z_n\}$ is Cauchy in V, and, since V is complete, it will converge to an element $z \in V$ such that $\|z\|_V < 1$ and we will also have $T(z) = y$. This proves (2.7).

More generally, if $r > 0$, there exists $s > 0$ such that

$$B_W(0; s) \ \subset \ T(B_V(0; r)).$$

Step 3: Let G be an open set in V. We now show that $T(G)$ is open, which will prove (ii). Let $y \in T(G)$. Then, there exists $x \in G$ such that $Tx = y$. Since G is open, there exists $r > 0$ such that $x + B_V(0; r) \subset G$. Hence, $y + T(B_V(0; r)) \subset T(G)$. But by Step 2, there exists $s > 0$ such that $B_W(0; s) \subset T(B_V(0; r))$ and so $y + B_W(0; s) \subset T(G)$ which means that $T(G)$ is open. This completes the proof that (i) $\Rightarrow$ (ii). ■

Proof that (ii) $\Rightarrow$ (iii)

Assume that the open mapping theorem is true. Then if $T : V \to W$ is a bijective and continuous linear map between the Banach spaces V and W, then it maps open sets onto open sets. This immediately implies that $T^{-1} : W \to V$ is continuous since, if $U \subset V$ is an open set, then $(T^{-1})^{-1}(U) = T(U)$ which is open in W. ■

Proof that (iii) $\Rightarrow$ (iv)

Let V and W be Banach spaces and let $T : V \to W$ be a linear map such that its graph, $G(T)$, is closed in $V \times W$. Define, for $x \in V$,

$$\|x\|_T \ = \ \|x\|_V + \|Tx\|_W.$$

Then it is easy to see that $\|\cdot\|_T$ defines a norm on V. If $\{x_n\}$ is a Cauchy sequence with respect to this norm, then it is also a Cauchy sequence with respect to $\|\cdot\|_V$ and $\{Tx_n\}$ is a Cauchy sequence in W. Thus, let $x_n \to x$ in V and let $Tx_n \to y$ in W. Since $G(T)$ is closed it follows that $y = Tx$ and so $\|x_n - x\|_T \to 0$. Thus V is complete with respect to the norm $\|\cdot\|_T$ as well. Further, since $\|x\|_V \leq \|x\|_T$, for every $x \in V$, the identity map from $(V, \|\cdot\|_T)$ onto $(V, \|\cdot\|_V)$ is a continuous bijection and hence the identity map in the opposite direction is also continuous, by (iii). Thus, there exists a constant $C > 0$ such that, for every $x \in V$, we have $\|x\|_T \leq C\|x\|_V$. In particular, we have that, for every $x \in V$,

$$\|Tx\|_W \;\leq\; C\|x\|_V.$$

Thus T is continuous. ■

Proof that (iv) $\Rightarrow$ (v)

Let V be a vector space which is complete with respect to both the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If $I : V \to V$ is the identity mapping, then the given inequality (2.2) implies that if $\|x_n - x\|_1 \to 0$ and if $\|x_n - y\|_2 \to 0$, then $y = x$. Thus the graph of the identity map from $(V, \|\cdot\|_1)$ onto $(V, \|\cdot\|_2)$ is closed and so the map is continuous, by the closed graph theorem. SInce the identity map the other way around is already given to be continuous, thanks to (2.2), we have that the norms are equivalent. ■

Proof that (v) $\Rightarrow$ (i).

Let $\{T_i\}_{i\in I}$ be a collection of continuous linear maps from a Banach space V to a normed linear space W such that (2.1) holds for all $x \in V$. Now consider the norm on V defined by

$$\|x\|_2 \;=\; \|x\|_V + \sup_{i\in I}\|T_i x\|_W,\; x \in V$$

which is well-defined in view of (2.1). If $\{x_n\}$ is a Cauchy sequence in V with respect to this norm, then it is also a Cauchy sequence with the original norm and so, $x_n \to x$ in V (with respect to the original norm). From the definition of $\|\cdot\|_2$, it follows that the sequences $\{T_i x_n\}$ are uniformly Cauchy from which we immediately see that

$$\sup_{i\in I}\|T_i x_n - T_i x\|_W \;\stackrel{n\to\infty}{\longrightarrow}\; 0.$$

Thus $\{x_n\}$ converges to x with respect to the norm $\|\cdot\|_2$ as well and so the space V is complete with respect to both norms. Since we have that $\|x\|_V \leq \|x\|_2$ for all $x \in V$, it follows from (iv) that these norms are equivalent. Thus there exists $C > 0$ such that for all $x \in V$, we have $\|x\|_2 \leq C\|x\|_V$. In particular, we have

$$\sup_{i\in I} \|T_i x\|_W \ \leq \ C\|x\|_V$$

for all $x \in V$ which implies that $\|T_i\| \leq C$ for every $i \in I$. The proof of Theorem 2.1 is thus complete. ■

Remark 2.1 The first step in the proof of the implication (i) $\Rightarrow$ (ii) occurs, in a more general form, in Schechter [4]. The remaining steps of this implication and the proof of the implication (ii) $\Rightarrow$ (iii), can be found in almost any standard text on functional analysis. For instance, see Kesavan [1]. The proofs of the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are also standard, though usually statement (v) is proved before (iv). But the proofs are essentially the same. The author learnt of the proof of the implication (v) $\Rightarrow$ (i) from a note by Ramaswamy and Ramasamy [3]. ■

Remark 2.2 In the proof of the implication (i) $\Rightarrow$ (ii),the first step uses the completeness of W, while the second step uses the completeness of V. The completeness of both the spaces is needed in Step 3 to complete the proof. ■

Remark 2.3 In the earlier version of this article (cf. Kesavan [2]), the closed graph theorem was the starting point and in an attempt to close the loop of the various implications, the proof that the uniform boundedness principle implies the closed graph theorem needed the reflexivity of the target space. In the above mentioned article, this implication was proved in the context of Hilbert spaces. The case where W is a reflexive Banach space is treated by Ramaswamy and Ramasamy [3]. Here, we have started with the open mapping theorem and have been able to close the loop without extra hypotheses. In particular, without extra hypotheses, we have also established that the uniform boundedness theorem implies the closed graph theorem. ■

Remark 2.4 Finally here is a quick proof that the closed graph theorem implies the open mapping theorem, without going through the uniform boundedness theorem. A proof of this was already given in Kesavan [2], but this is even simpler.

- If $T : V \to W$ is a continuous bijection between Banach spaces, then it is immediate to check that the graph of T^{-1} is closed. Thus T^{-1} is also continuous. This proves the implication (iv) $\Rightarrow$ (iii).

- Let V be a normed linear space and let W be a closed subspace. Consider the canonical mapping $\pi : V \to V/W$, given by $x \mapsto x + W$. This is continuous, since $\|x + W\|_{V/W} \leq \|x\|_V$, by definition. Further, if $\|x + W\|_{V/W} < r$, there exists $w \in W$ such that $\|x + w\|_V < r$ and we also have $\pi(x + w) = \pi(x) = x + W$. Thus $B_{V/W}(0; r) \subset \pi(B_V(0; r))$, from which we easily deduce that π is also an open map (cf. Step 3 of the proof of the implication (i) $\to$ (ii) of Theorem 2.1).

- Let $T : V \to W$ be a surjective and continuous linear map between the Banach spaces V and W. Let $Z = \mathrm{Ker}(T)$. Define $\overline{T} : V/Z \to W$ by $\overline{T}(x + Z) = Tx$. It is easy to see that this map is well-defined and that it is a bijection. It is continuous since, for every $z \in Z$, we have $Tx = T(x + z)$ and so we deduce that

  $$\|\overline{T}(x + Z)\|_W \; = \; \|Tx\|_W \; \leq \; \|T\| \inf_{z \in Z} \|x + z\| \; = \; \|T\|\|x + Z\|_{V/Z}.$$

  Thus, by (iii), it is an isomorphism, and hence is an open map. Now $T = \overline{T} \circ \pi$ and so T is an open map as well. This proves (ii). ■

We have thus seen that the uniform boundedness theorem, the open mapping theorem and the closed graph theorem are all equivalent to each other and the only common ingredient in all of the proofs is the completeness of the spaces involved. We will now reproduce the proof (due to Sokal [5]), presented in Kesavan [2], of the uniform boundedness theorem, which does not use the Baire category theorem. Thus all the three grand theorems can be proved without using Baire's theorem.

Proof of the uniform boundedness theorem:

Step 1. Let $x, y \in V$. Since $2y = (x + y) - (x - y)$, the triangle inequality gives

$$\|Ty\|_W \; \leq \; \frac{1}{2}[\|T(x+y)\|_W + \|T(x-y)\|_W] \; \leq \; \max\{\|T(x+y)\|_W, \|T(x-y)\|_W\}$$

where T is a continuous linear map from V to W. If we now take the supremum as y varies over the open ball centred at the origin and of radius $r > 0$, we deduce that

$$\sup_{x' \in B_V(x;r)} \|Tx'\|_W \ \geq \ r\|T\|. \tag{2.8}$$

Step 2. Assume that $\{\|T_i\|\}_{i\in I}$ is unbounded. Then choose a sequence $\{T_n\}_{n=1}^\infty$ from this family such that $\|T_n\| \geq 4^n$ for each n. Set $x_0 = 0$. Then, by Step 1, we can inductively find $\{x_n\}_{n=1}^\infty$ in V such that $\|x_n - x_{n-1}\|_V < 3^{-n}$ and $\|T_n x_n\|_W > \frac{2}{3}3^{-n}\|T_n\|$. By construction, since $\sum 3^{-n}$ is convergent, the sequence $\{x_n\}$ is Cauchy and since V is complete, we have $x_n \to x$ in V.

Now, if $m > n$, we have

$$\|x_n - x_m\|_V \ \leq \ 3^{-(n+1)} + 3^{-(n+2)} + \cdots + 3^{-m}.$$

Keeping n fixed and letting $m \to \infty$, we deduce that

$$\|x_n - x\|_V \ \leq \ \frac{1}{2}3^{-n}.$$

Then, by the triangle inequality, we get that

$$\|T_n x\|_W \ \geq \ \|T_n x_n\|_W - \|T_n(x_n - x)\|_W \ \geq \ \frac{1}{6}3^{-n}\|T_n\| \ \geq \ \frac{1}{6}\left(\frac{4}{3}\right)^n.$$

Thus $\{\|T_n x\|_W\}$ is unbounded which contradicts (2.1). This completes the proof. ■

Remark 2.4 The above proof just proves the uniform boundedness theorem to the extent that pointwise boundedness implies uniform boundedness. However, when proved using the Baire category theorem, we can also make statements on what happens when we do not have pointwise boundedness. This has nice applications, in particular, to the question of divergence of Fourier series. See, for instance, Kesavan [1] for details. ■

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