Integration and Polar Coordinates

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Abstract: A simple proof of the formula for the integration of radial functions on $\mathbb{R}^N, N \geq 2$, is given. As an application, the volume of the N-dimensional unit ball is computed.

1 Introduction

The method of substitution, also known as change of variables, is a useful tool in integration. One of the high points in a course on multivariate calculus is the change of variable formula which states that if $f = (f_1, f_2, \cdots, f_N) : E \subset \mathbb{R}^N \to \mathbb{R}^N$ is a bijection which is smooth both ways, and if g is an integrable function on $f(E)$, then

$$\int_{f(E)} g(x)\ dx \ = \ \int_E g(f(x))|\det(f'(x))|\ dx$$

where $x = (x_1, x_2, \cdots, x_N)$, $f'(x)$ is the Jacobian matrix at x whose ij-th entry is given by

$$\frac{\partial f_i}{\partial x_j}(x)$$

and dx indicates that we are performing an N-dimensional volume integration, *i.e.*, we have an N-fold integral with respect to $dx_1 dx_2 \cdots dx_N$.

The most familiar application of this is the case of polar coordinates in the plane given by $x = r\cos\theta$ and $y = r\sin\theta$ and the integral

$$\int\int g(x,y)\ dxdy$$

transforms as

$$\int\int \widetilde{g}(r,\theta)\ rdrd\theta$$

with appropriate limits of integration (here $g(x,y) = \widetilde{g}(r,\theta)$). Similarly, in $\mathbb{R}^3$, we have the spherical polar coordinates $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ and the integral

$$\int\int\int g(x,y,z)\ dxdydz$$

transforms as

$$\int\int\int \widetilde{g}(r,\theta,\varphi)\ r^2\sin\theta drd\theta d\varphi,$$

again, with appropriate limits of integration, where $g(x,y,z) = \widetilde{g}(r,\theta,\varphi)$.

Polar coordinates are particularly useful when the function to be integrated is a radial function, *i.e.* it depends only on

$$r \ = \ |x| \ = \ \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}.$$

In such a case, the integral reduces to one involving just a single variable. For instance, if we are integrating over the entire space, the integral reduces to

$$2\pi \int_0^\infty \widetilde{g}(r) r \; dr$$

when $N = 2$ and to

$$4\pi \int_0^\infty \widetilde{g}(r) r^2 \; dr$$

when $N = 3$. The most familiar example of this is the evaluation of the Gaussian integral, ubiquitous in probability theory,

$$\int_{-\infty}^\infty e^{-x^2} dx.$$

If we set this as I, then,

$$I^2 \;=\; \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} \; dxdy \;=\; 2\pi \int_0^\infty e^{-r^2} r \; dr$$

which yields $I = \sqrt{\pi}$.

When the dimension exceeds three, it is difficult to explicitly write down the spherical coordinates and even if we succeed, the ensuing calculations involving the N-dimensional Jacobian will certainly be horrendous. We would like to have a simple formula, at least for the integration of radial functions, and this is what we will set out to establish in this article.

Looking at the cases $N = 2$ and $N = 3$, we observe that $2\pi r$ is the perimeter of the circle of radius r and $4\pi r^2$ is the surface area of the sphere of radius r in these respective dimensions and so we are tempted to conjecture that the integral of a radial function would transform as the integral with respect to r after multiplication by the $(N-1)$-dimensional surface measure of the sphere of radius r in $\mathbb{R}^N$. Is this correct, and, if so, what is the surface measure of the sphere in $\mathbb{R}^N$?

Let us denote by ω_N the volume of the unit ball in $\mathbb{R}^N$ and by α_N, the $(N-1)$-dimensional surface measure of the unit sphere. The change of variables $x \mapsto x/R$ maps the ball of radius R onto the unit ball and the Jacobian turns out to be a scalar matrix and so we easily conclude that the volume of the ball of radius R is $\omega_N R^N$ from the change of variable formula mentioned at the beginning. What about the surface measure of the sphere of radius R? For this we have a very nice application of Gauss' divergence theorem: if $\Omega \subset \mathbb{R}^N$ is a domain, $\partial\Omega$ its boundary on which we have a well-defined outward unit normal ν, then for a smooth vector valued function $v : \Omega \to \mathbb{R}^N$, we have

$$\int_\Omega \mathrm{div}(v(x))\ dx \ = \ \int_{\partial\Omega} v(x).\nu(x)\ dS(x)$$

where the 'dot' denotes the usual scalar product in $\mathbb{R}^N$ and $dS(x)$ indicates that we are performing a surface integral with respect to the appropriate induced surface measure.

Setting $v(x) = x$, we get $\mathrm{div}(v(x)) = N$ and so the left-hand side just gives N times the volume of the domain. If Ω is the ball of radius R, then it is $N\omega_N R^N$. The unit outer normal at a point x on the boundary $\partial\Omega$ of the ball Ω of radius R and centre at the origin is just x/R and so $v(x).\nu(x)$ reduces to R and so the right-hand side is just R times the surface measure of $\partial\Omega$. Hence we conclude that the surface measure of the sphere of radius R is $N\omega_N R^{N-1}$, which when $R = 1$ yields

$$\alpha_N \ = \ N\omega_N$$

.

In the sequel, we will prove that the integral of a radial function $f(x) = \widetilde{f}(r)$ over a region which is spherically symmetric about the origin is indeed given by

$$N\omega_N \int \widetilde{f}(r) r^{N-1}\ dr \ = \ \alpha_N \int \widetilde{f}(r) r^{N-1}\ dr$$

with appropriate limits of integration. We will also determine ω_N.

2 Integration of radial functions

Let Ω be the ball of radius $R > 0$ with centre at the origin. Consider a continuous real valued function, f, on the closed ball which is radial, *i.e.* $f(x) = \widetilde{f}(|x|)$ where $\widetilde{f} : [0, R] \to \mathbb{R}$ is a continuous function. Consider a partition

$$\mathcal{P} : 0 = r_0 \ < \ r_1 \ < \ r_2 \ < \ \cdots \ < \ r_n = R$$

of the interval $[0, R]$. Let $\Omega_i,\ 1 \leq i \leq n$, denote the annular region

$$\{x \in \mathbb{R}^N \mid r_{i-1} \leq |x| < r_i\}$$

so that $\Omega = \cup_{i=1}^n \Omega_i$. By the mean value theorem, there exists $\xi_i \in (r_{i-1}, r_i)$ such that

$$r_i^N - r_{i-1}^N \;=\; N\xi_i^{N-1}(r_i - r_{i-1}).$$

Choose $y_i \in \Omega_i$ such that $|y_i| = \xi_i$. Define

$$f_{\mathcal{P}}(x) \;=\; f(y_i),\; x \in \Omega_i,\; 1 \le i \le n.$$

Let

$$\mu(\mathcal{P}) \;=\; \max_{1\le i\le n}(r_i - r_{i-1}).$$

If $x \in \Omega_i$, then

$$|f(x) - f_{\mathcal{P}}(x)| \;=\; |\widetilde{f}(|x|) - \widetilde{f}(|y_i|)| \;=\; |\widetilde{f}(|x|) - \widetilde{f}(\xi_i)|.$$

Since $\widetilde{f}$ is uniformly continious, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in \Omega$, we have

$$|f(x) - f_{\mathcal{P}}(x)| \;<\; \varepsilon$$

for all partitions $\mathcal{P}$ such that $\mu(\mathcal{P}) < \delta$. Thus, $f_{\mathcal{P}}$ converges uniformly to f as $\mu(\mathcal{P}) \to 0$. Consequently,

$$\int_\Omega f_{\mathcal{P}}(x)\;dx \;\to \int_\Omega f(x)\;dx$$

as $\mu(\mathcal{P}) \to 0$. But since $f_{\mathcal{P}}$ is constant in each subdomain Ω_i and since the volume of Ω_i is given by $\omega_N(r_i^N - r_{i-1}^N)$, we get

$$\int_\Omega f_{\mathcal{P}}(x)\;dx \;=\; \sum_{i=1}^n \widetilde{f}(\xi_i)\omega_N(r_i^N - r_{i-1}^N) \;=\; \sum_{i=1}^n \widetilde{f}(\xi_i)\omega_N N\xi_i^{N-1}(r_i - r_{i-1})$$

and the last term converges, as $\mu(\mathcal{P}) \to 0$, to

$$\int_0^R N\omega_N \widetilde{f}(r) r^{N-1}\;dr.$$

Thus, we have shown that if f is a continuous and radial function (*i.e.* $f(x) = \widetilde{f}(|x|)$) on the closed ball of radius R, centered at the origin, then

$$\int_\Omega f(x)\;dx \;=\; N\omega_N \int_0^R \widetilde{f}(r) r^{N-1}\;dr.$$

By passing to the limit as $R \to \infty$, it is now a simple exercise to see that if f is a continuous and improperly Riemann integrable function on $\mathbb{R}^N$ which is radial, then

$$\int_{\mathbb{R}^N} f(x)\, dx \;=\; N\omega_N \int_0^\infty \widetilde{f}(r) r^{N-1}\, dr.$$

3 Volume of the unit ball

Let us now apply the preceding result to the function

$$f(x) \;=\; e^{-\sum_{i=1}^N x_i^2} \;=\; e^{-r^2}.$$

On one hand (see Section 1),

$$\int_{\mathbb{R}^N} f(x)\, dx \;=\; \Pi_{i=1}^N \int_{-\infty}^\infty e^{-x_i^2}\, dx_i \;=\; \pi^{\frac{N}{2}}.$$

On the other hand, using polar coordinates,

$$\int_{\mathbb{R}^N} f(x)\, dx \;=\; N\omega_N \int_0^\infty e^{-r^2} r^{N-1}\, dr.$$

Setting $r^2 = s$, we get

$$\int_0^\infty e^{-r^2} r^{N-1}\, dr \;=\; \frac{1}{2}\int_0^\infty e^{-s} s^{\frac{N}{2}-1}\, ds \;=\; \frac{1}{2}\Gamma\left(\frac{N}{2}\right)$$

where Γ denotes the usual Gamma function defined by

$$\Gamma(t) \;=\; \int_0^\infty e^{-s} s^{t-1}\, ds,\;\; t > 0.$$

By a simple integration by parts, we see that

$$\Gamma(t+1) \;=\; t\Gamma(t),\;\; t > 0 \tag{1}$$

and so we deduce from the above computations that

$$\omega_N \;=\; \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})} \;=\; \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}.$$

When $t = 1/2$, the integral defining the Gamma function reduces to the Gaussian integral mentioned at the beginning of this article and we have

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

By repeated use of (1) we see that for any positive integer n, $\Gamma(n+1) = n!$. Thus we can compute $\Gamma(\frac{N}{2}+1)$ by appealing to these facts and repeatedly using (1).

Examples

$$\omega_2 = \frac{\pi}{\Gamma(2)} = \pi.$$

$$\omega_3 = \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2}+1)}.$$

Now,

$$\Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

Thus,

$$\omega_3 = \frac{4}{3}\pi.$$

In the same way we can show that

$$\omega_4 = \frac{\pi^2}{2},\ \omega_5 = \frac{8\pi^2}{15}$$

and so on.

4 Concluding remarks

The integration formula for continuous radial functions that was proved in this article is a very special case of a deep result in measure theory called the **coarea formula**. It states that if $u : \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz continuous function and is such that for almost every $r \in \mathbb{R}$, the *level set*

$$\{u = r\} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \mid u(x) = r\}$$

is a smooth $(N-1)$-dimensional hypersurface in $\mathbb{R}^N$, and if $f : \mathbb{R}^N \to \mathbb{R}$ is a continuous and integrable function, then

$$\int_{\mathbb{R}^N} f(x)|Du(x)|\ dx \ = \ \int_{-\infty}^{\infty} \left(\int_{\{u=r\}} f(x)\ dS(x) \right)\ dr.$$

(By another deep theorem of Radamacher, Lipschitz continuous functions are differentiable almost everywhere and Du denotes the gradient vector of u.) In particular, choosing $u(x) = |x|$, we get the polar coordinate formula

$$\int_{\mathbb{R}^N} f(x)\ dx \ = \ \int_0^{\infty} \left(\int_{\{|x|=r\}} f(x)\ dS(x) \right)\ dr.$$

If f is radial, then, following our usual notation, the inner integral just becomes $\widetilde{f}(r)$ times the surface measure of the sphere of radius r which gives $N\omega_N r^{N-1}\widetilde{f}(r)$. For a detailed treatment of the coarea (pronounced co-area) formula, the reader is referred to the treatise 'Measure Theory and Fine Properties of Functions' by L. C. Evans and R. F. Gariepy (CRC Press, 1992).

Finally, we can also derive a recursive relation for the volume of the unit ball. Indeed, we can slice up the N-dimensional hemisphere by sections parallel to the base. Then at height t, $0 < t < 1$, we get an infinitesimal 'cylinder' with 'base' an $(N-1)$-dimensional ball of radius $\sqrt{1-t^2}$ and of 'height' dt. Thus,

$$\omega_N \ = \ 2\int_0^1 \omega_{N-1}(1-t^2)^{\frac{N-1}{2}}\ dt \ = \ 2\omega_{N-1}\int_0^{\frac{\pi}{2}} \cos^N\theta\ d\theta.$$

Recall that

$$\int_0^{\frac{\pi}{2}} \cos^N\theta\ d\theta \ = \ \begin{cases} \frac{N-1}{N}\frac{N-3}{N-2}\cdots\frac{1}{2}\frac{\pi}{2}, & \text{if } N \text{ is even} \\ \frac{N-1}{N}\frac{N-3}{N-2}\cdots\frac{2}{3}, & \text{if } N \text{ is odd.} \end{cases}$$

The reader is referred to an article by K. B. Athreya (cf. *Resonance*, April 2008) where this idea is developed and the asymptotic behaviour of ω_N, as $N \to \infty$ is examined.