Continuous functions that are nowhere differentiable

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Abstract

It is shown that the existence of continuous functions on the interval $[0, 1]$ that are nowhere differentiable can be deduced from the Baire category theorem. This approach also shows that there is a preponderance of such functions.

1 Introduction

The French mathematician Hermite, in a letter written to Stieltjes, dated May 20, 1893, wrote ‘I turn away with fear and horror from the lamentable plague of continuous functions which do not have derivatives ...’(cf. Pinkus [6]). The earliest universally acknowledged explicit example of a continuous function which is nowhere differentiable is due to Weierstrass (1872) given by

$$\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $ab > 1 + \frac{3}{2}\pi$. It is also said that Bolzano constructed such an example (in the 1830s), which was not published. Since then a number of variants of Weierstrass’ example have appeared in the literature. Here are some of them.

- $$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin(3^n x).$$

- (cf. Hardy [3])
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 \pi x).$$

- (cf. Rudin [7]) Define
$$\varphi(x) \;=\; \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2 \end{cases}$$
and extend it to all of $\mathbb{R}$ by setting $\varphi(x+2) = \varphi(x)$. Then the function defined by the series
$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$
is again continuous and nowhere differentiable.

In the above three examples, the series are clearly uniformly convergent by the Weierstrass M-test and so the sum defines a continuous function. One has to show that it is nowhere differentiable.

Another type of example is constructed as follows. Consider the space $\mathcal{C}[0,1]$ (the space of continuous functions on $[0,1]$) with the usual norm topology generated by the norm

$$\|f\|_\infty \;=\; \max_{x\in[0,1]} |f(x)|.$$

Let

$$X = \{f \in \mathcal{C}[0,1] \mid f(0) = 0,\ f(1) = 1\}.$$

Then it is a closed subset of $\mathcal{C}[0,1]$ and is hence a complete metric space in its own right. For $f \in X$, define

$$T(f)(x) \;=\; \begin{cases} \frac{3}{4}f(3x), & 0 \le x \le \frac{1}{3}, \\ \frac{1}{4} + \frac{1}{2}f(2-3x), & \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{1}{4} + \frac{3}{4}f(3x-2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Then it can be shown that T maps X into itself and that

$$\|T(f) - T(g)\|_\infty \;\le\; \frac{3}{4}\|f-g\|_\infty.$$

Hence, by the contraction mapping theorem, there exists $h \in X$ such that $T(h) = h$. It can be shown then that h is nowhere differentiable.

The aim of the present article is to show the existence of continuous but nowhere differentiable functions, without exhibiting one. The proof, following the ideas of Banach [1] and Mazurkiewicz [5], uses the Baire category theorem which can be stated as follows.

Theorem 1.1 *(*Baire*)Let X be a complete metric space. If $\{U_n\}_{n=1}^\infty$ is a sequence of open and dense sets in X, then*

$$\cap_{n=1}^\infty U_n$$

is also dense in X. ■

Equivalently, a complete metric space cannot be the countable union of a family of closed and nowhere dense sets. In technical parlance, a complete metric space is said to be of the 'second category' (the first category being topological spaces which are countable unions of closed and nowhere dense sets), and hence the word 'category' in the name of the theorem. For a proof, see any text on functional analysis (for instance, see Ciarlet [2] or Kesavan [4]).

Baire's theorem is the corner stone of the famous trinity of theorems in functional analysis, *viz.* the uniform boundedness principle, the open mapping theorem and the closed graph theorem. As a consequence of the uniform boundedness principle, we can show that for a large class of continuous functions, the Fourier series diverges on a large set of points (see, for instance, Kesavan [4]).

We will use Baire's theorem to prove the existence of nowhere differentiable functions in $\mathcal{C}[0,1]$. This approach also shows that the class of such functions is quite large. Our presentation is an adaptation of that found in Ciarlet [2].

2 Approximation by smooth functions

A celebrated theorem of Weierstrass states that any continuous function on $[0,1]$ can be uniformly approximated by polynomials. To make this presentation as self-contained as possible, we will prove a slightly weaker result which is enough for our purposes, *viz.* that any continuous function on $[0,1]$ can be uniformly approximated by smooth functions.

Consider the function

$$\rho(x) \ = \ \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

It is not difficult to see that this defines a $\mathcal{C}^\infty$ function on $\mathbb{R}$ whose support is the closed ball centered at the origin and with unit radius. For $\varepsilon > 0$, define

$$\rho_\varepsilon(x) \ = \ (k\varepsilon)^{-1}\rho\left(\frac{x}{\varepsilon}\right)$$

where

$$k \ = \ \int_{-\infty}^{\infty} \rho(x)\ dx \ = \ \int_{-1}^{1} \rho(x)\ dx.$$

Then, it is easy to see that ρ_ε is also $\mathcal{C}^\infty$ and its support is the closed ball centered at the origin with radius ε. Further

$$\int_{-\infty}^{\infty} \rho_\varepsilon(x)\ dx \ = \ \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(x)\ dx \ = \ 1.$$

Recall that if f and g are continuous real-valued functions defined on $\mathbb{R}$, with one of them having compact support, the convolution product $f * g$ defined by

$$(f * g)(x) \; = \; \int_{-\infty}^{\infty} f(x-y)g(y) \; dy \; = \; \int_{-\infty}^{\infty} g(x-y)f(y) \; dy$$

is well defined and is a continuous function. Further, if one of them is in $\mathcal{C}^k(\mathbb{R})$, then $f * g \in \mathcal{C}^k(\mathbb{R})$ for any $1 \leq k \leq \infty$. If $\mathrm{supp}(F)$ denotes the support of a function F, then

$$\mathrm{supp}(f * g) \; \subset \; \mathrm{supp}(f) + \mathrm{supp}(g)$$

where, for subsets A and B of $\mathbb{R}$, we define

$$A + B \; = \; \{x + y \mid x \in A, y \in B\}.$$

Proposition 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Then $\rho_\varepsilon * f$ converges uniformly to f as $\varepsilon \to 0$.*

Proof: Let K be the support of f. Then K is a compact subset of $\mathbb{R}$. Without loss of generality, we can assume $0 < \varepsilon < 1$ so that $\rho_\varepsilon * f$ is a $\mathcal{C}^\infty$ function with support contained in the fixed compact set

$$\{x \in \mathbb{R} \mid |x| \leq 1\} + K.$$

Clearly f is uniformly continuous and so, given $\eta > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| < \delta$. Now, since the integral of ρ_ε is unity, we can write

$$(\rho_\varepsilon * f)(x) - f(x) \; = \; \int_{-\varepsilon}^{\varepsilon} (f(x-y) - f(x))\rho_\varepsilon(y) \; dy.$$

Thus, if $\varepsilon < \delta$ then

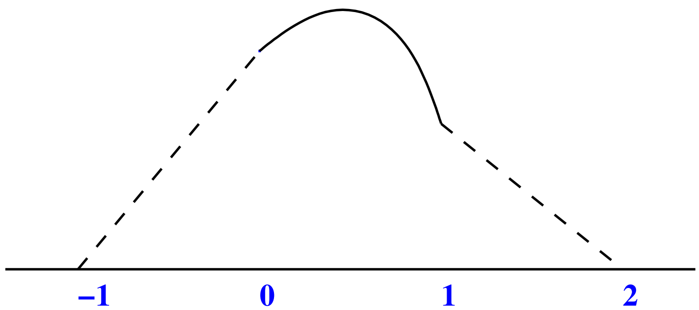
$$|(\rho_\varepsilon * f)(x) - f(x)| \; \leq \; \int_{-\varepsilon}^{\varepsilon} |f(x-y) - f(x)|\rho_\varepsilon(y) \; dy \; \leq \eta$$

for all x and this completes the proof. ■

Corollary 2.1 *Let $f \in \mathcal{C}[0,1]$. Then f can be uniformly approximated by smooth functions.*

Proof: Given $f \in \mathcal{C}[0,1]$, we can extend it to a continuous function with compact support in $\mathbb{R}$. For example, define

$$\widetilde{f}(x) \;=\; \begin{cases} 0, & \text{if } x < -1 \text{ or if } x > 2, \\ (x+1)f(0), & \text{if } x \in [-1,0], \\ f(x), & \text{if } x \in [0,1], \\ (2-x)f(1), & \text{if } x \in [1,2]. \end{cases}$$



Now $\widetilde{f}$ can be uniformly approximated by smooth functions in $\mathbb{R}$ and so their restrictions to $[0,1]$ will approximate f uniformly on $[0,1]$. ■

Proposition 2.2 *Let $f \in \mathcal{C}[0,1]$. Let $\varepsilon > 0$ and n, a positive integer, be given. Then there exists a piecewise linear continuous function g, defined on $[0,1]$ such that $\|f - g\|_\infty < \varepsilon$ and such that $|g'(t)| > n$ at all points where the derivative exists.*

Proof: In view of the corollary above, we can assume that f is a smooth function defined on $[0,1]$.
Step 1. Since f is smooth, f' is bounded in $[0,1]$. Let $|f'(x)| \leq M$ for all $x \in [0,1]$. Since f is continuous on $[0,1]$, it is uniformly continuous and so there exists $\delta > 0$ such that, whenever $|x-y| < \delta$, we have $|f(x) - f(y)| < \frac{\varepsilon}{4}$. Now, choose $h > 0$ such that

$$h \;<\; \min\left\{\delta, \frac{\varepsilon}{2(M+n)}\right\}.$$

Step 2. Now choose a partition

$$\mathcal{P} : 0 \;=\; t_0 \;<\; t_1 \;<\; \cdots \;<\; t_k \;=1$$

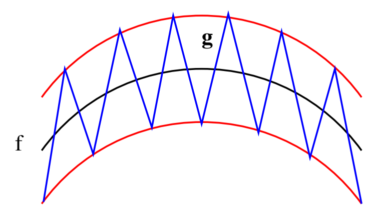
such that

$$\max_{0 \leq i \leq k-1} (t_{i+1} - t_i) \;\leq\; h.$$

Let $g : [0,1] \to \mathbb{R}$ be a piecewise linear and continuous function, defined on each sub-interval $[t_i, t_{i+1}], 0 \le i \le k-1$, as follows:

$$\begin{aligned} g(t_i) &= f(t_i) + (-1)^i \tfrac{\varepsilon}{4}, \\ g(t_{i+1}) &= f(t_{i+1}) + (-1)^{i+1} \tfrac{\varepsilon}{4}, \\ g(t) &= \tfrac{t_{i+1}-t}{t_{i+1}-t_i} g(t_i) + \tfrac{t-t_i}{t_{i+1}-t_i} g(t_{i+1}),\ t_i < t < t_{i+1}. \end{aligned}$$

The function g is differentiable except at the points $\{t_1, \cdots, t_{k-1}\}$.



Step 3. For $t \in [t_i, t_{i+1}],\ 0 \le i \le k-1$, we have

$$g(t) - f(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i}(g(t_i) - f(t)) + \frac{t-t_i}{t_{i+1}-t_i}(g(t_{i+1}) - f(t))$$

so that

$$|g(t) - f(t)| \le |f(t_i) - f(t)| + |f(t_{i+1}) - f(t)| + \frac{\varepsilon}{2} < \varepsilon$$

since $|t - t_i|$ and $|t - t_{i+1}|$ are both less than, or equal to $h < \delta$. Thus, it follows that $\|f - g\|_\infty < \varepsilon$.

Step 4. For any $t \in (t_i, t_{i+1}),\ 0 \le i \le k-1$, we have

$$g'(t) = \frac{f(t_{i+1}) - f(t_i) + (-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1}-t_i} = f'(\xi_i) + \frac{(-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1}-t_i}$$

where $\xi_i \in (t_i, t_{i+1})$. Thus, by our choice of h, we have

$$\begin{aligned} |g'(t)| &= \left| \tfrac{(-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1}-t_i} + f'(\xi_i) \right| \\ &\ge \tfrac{\varepsilon}{2(t_{i+1}-t_i)} - |f'(\xi_i)| \\ &\ge \tfrac{\varepsilon}{2h} - M \\ &> n \end{aligned}$$

which completes the proof. ■

3 The main result

Proposition 3.1 *Let $f \in \mathcal{C}[0,1]$ be differentiable at some point $a \in [0,1]$. Then, there exists a positive integer N such that*

$$\sup_{h\neq 0}\left|\frac{f(a+h)-f(a)}{h}\right| \;\leq\; N.$$

Proof: Since f is differentiable at $a \in [0,1]$, there exists $h_0 > 0$ such that for all $0 < |h| \leq h_0$, we have

$$\left|\frac{f(a+h)-f(a)}{h} - f'(a)\right| \;\leq\; 1.$$

Thus, for all $0 < |h| \leq h_0$, we have

$$\left|\frac{f(a+h)-f(a)}{h}\right| \;\leq\; 1 + |f'(a)|.$$

If $|h| \geq h_0$, then trivially

$$\left|\frac{f(a+h)-f(a)}{h}\right| \;\leq\; \frac{2\|f\|_\infty}{h_0}.$$

Thus we only need to take

$$N \geq \max\left\{1 + |f'(a)|, \frac{2\|f\|_\infty}{h_0}\right\}. \;\blacksquare$$

Let us now define, for each positive integer n,

$$\mathcal{A}_n \;=\; \left\{f \in \mathcal{C}[0,1] \mid \sup_{h\neq 0}\left|\frac{f(a+h)-f(a)}{h}\right| \leq n \text{ for some } a \in [0,1]\right\}.$$

Proposition 3.2 *For each positive integer n, the set $\mathcal{A}_n$ is closed in $\mathcal{C}[0,1]$.*

Proof: Let $\{f_k\}$ be a sequence in $\mathcal{A}_n$ such that $f_k \to f$ in $\mathcal{C}[0,1]$. Then, there exists a sequence $\{a_k\}$ in $[0,1]$ such that, for each k,

$$\sup_{h\neq 0}\left|\frac{f_k(a_k+h)-f_k(a_k)}{h}\right| \;\leq\; n.$$

Let $\{a_{k_l}\}$ be a convergent subsequence, converging to $a \in [0,1]$.

Let $h \neq 0$ be given. Choose h_{k_l} such that $a_{k_l} + h_{k_l} = a + h$.Thus the sequence $\{h_{k_l}\}$ converges to $h \neq 0$ and so we may assume, without loss of generality, that it is a sequence of non-zero real numbers. Now

$$|f(a+h) - f_{k_l}(a_{k_l} + h_{k_l})| \;=\; |(f - f_{k_l})(a+h)| \;\leq\; \|f - f_{k_l}\|_\infty.$$

Also

$$|f(a) - f_{k_l}(a_{k_l})| \;\leq\; |f(a) - f(a_{k_l})| + |f(a_{k_l}) - f_{k_l}(a_{k_l})| \;\leq\; |f(a) - f(a_{k_l})| + \|f - f_{k_l}\|_\infty.$$

By the continuity of f and the convergence of $\{f_{k_l}\}$ to f, we then deduce that

$$\left|\frac{f(a+h) - f(a)}{h}\right| \;=\; \lim_{l\to\infty}\left|\frac{f_{k_l}(a_{k_l} + h_{k_l}) - f_{k_l}(a_{k_l})}{h_{k_l}}\right| \;\leq n$$

which shows that $f \in \mathcal{A}_n$ as well, which completes the proof. ■

Proposition 3.3 *For each positive integer n, the set $\mathcal{A}_n$ has empty interior.*

Proof: Given $\varepsilon > 0$, a positive integer n and a function $f \in \mathcal{A}_n$, let g be constructed as in the proof of Proposition 2.2. Then it is clear that the ball centered at f and of radius ε in $\mathcal{C}[0,1]$ contains g and that $g \notin \mathcal{A}_n$. This completes the proof. ■

We can now prove the main theorem.

Theorem 3.1 *There exist continuous functions on the interval $[0,1]$ which are nowhere differentiable. In fact the collection of all such functions forms a dense subset of $\mathcal{C}[0,1]$.*

Proof: By Baire's theorem and the two preceding propositions, it follows that

$$\mathcal{C}[0,1] \;\neq\; \cup_{n=1}^\infty \mathcal{A}_n.$$

From the definition of the sets $\mathcal{A}_n$ and from Proposition 3.1, it follows that every function in

$$\mathcal{C}[0,1] \backslash \cup_{n=1}^\infty \mathcal{A}_n \;=\; \cap_{n=1}^\infty (\mathcal{C}[0,1] \backslash \mathcal{A}_n)$$

is nowhere differentiable and also that this set is dense, since it is the countable intersection of open dense sets. ■

In particular, it follows that every continuous function on $[0,1]$, irrespective of its smoothness, is the uniform limit of functions that are nowhere differentiable!

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