A note on some approximation theorems in measure theory

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Abstract

Usual proofs of the density of continuous functions with compact support in the Lebesgue spaces $L^p(\Omega)$, where $1 \leq p < \infty$, and Ω is an open subset of $\mathbb{R}^N$, appeal to Lusin's theorem. It is shown here that this density theorem can be proven directly and that Lusin's theorem can be deduced from it.

1 Introduction

One of the important approximation theorems in the theory of measure and integration is the density of continuous functions with compact support in the Lebesgue spaces $L^p(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$ and $1 \leq p < \infty$. This result, together with the technique of convolution with mollifiers, shows that infinitely differentiable functions with compact support in Ω are also dense in $L^p(\Omega), 1 \leq p < \infty$. Further, separability properties of these spaces can also be deduced from this result. These are just some of the applications of this theorem.

Usual proofs of this approximation theorem appeal to Lusin's theorem which states that a measurable function on a set of finite Lebesgue measure agrees with a continuous function with compact support except possibly on a set whose measure is less than an arbitrarily small preassigned positive number.

In this note we will show that we can actually prove the density theorem by starting from simple results based on approximation of measurable functions by step functions and that we can deduce Lusin's theorem as a consequence.

Throughout the sequel, we will assume that $\mathbb{R}^N$ is equipped with the Lebesgue measure (denoted μ). Let $1 \leq p < \infty$. If $\Omega \subset \mathbb{R}^N$ is an open subset, then $L^p(\Omega)$ will denote the space (of equivalence classes, up to equality almost everywhere) of real (or complex) valued functions f defined on Ω with the property

$$\int_\Omega |f|^p \, dx \ < \ \infty.$$

The norm on this space is denoted $\|\cdot\|_p$ and is defined by

$$\|f\|_p \ = \ \left(\int_\Omega |f|^p \, dx\right)^{\frac{1}{p}}.$$

The essential supremum of an essentially bounded function f is denoted by $\|f\|_\infty$. The space of continuous functions with compact support in Ω will be denoted by $\mathcal{C}_c(\Omega)$ and will be equipped with the usual sup-norm, also denoted by $\|\cdot\|_\infty$.

By a **simple function**, we will mean a function of the form

$$f = \sum_{i=1}^{k} \alpha_i \chi_{E_i}, \tag{1.1}$$

where, for $1 \leq i \leq k$, the α_i are real (or complex) numbers and χ_{E_i} are the characteristic (also called indicator) functions of the measurable subsets $E_i \subset \Omega$.

By a **box** in $\mathbb{R}^N$, we will mean a set of the form

$$\Pi_{j=1}^N I_j$$

where the I_j are all finite intervals in $\mathbb{R}$. A closed box is the corresponding product of closed intervals and an open box is the product of open intervals. A half-open box is a box of the form

$$\Pi_{j=1}^N [a_j, b_j).$$

Lemma 1.1 *Every open set in $\mathbb{R}^N$ can be written as a countable disjoint union of half-open boxes.*

Proof: For a positive integer n, let $\mathcal{P}_n$ denote the collection of all points in $\mathbb{R}^N$ whose coordinates are all integral multiples of 2^{-n}. Let $\mathcal{B}_n$ denote the collection of all half-open boxes with each edge of length 2^{-n} and with vertices at the points of $\mathcal{P}_n$. The following conclusions are obvious by inspection:
(i) For a fixed n, each point $x \in \mathbb{R}^N$ belongs to exactly one box in $\mathcal{B}_n$.
(ii) If $Q \in \mathcal{B}_m, Q' \in \mathcal{B}_n$ where $n > m$, then either $Q' \subset Q$ or $Q \cap Q' = \emptyset$.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $x \in \Omega$. Then x lies in an open ball contained in Ω and so we can find a $Q \in \mathcal{B}_n$, for some sufficiently large n, containing x and which is contained inside this ball. In other words, Ω is the union of all boxes contained inside it and belonging to some $\mathcal{B}_n$. This collection of boxes is clearly countable but may not be disjoint. Now choose all those boxes in this collection which are in $\mathcal{B}_1$ and discard boxes of $\mathcal{B}_k, k \geq 2$, which lie inside these selected boxes. From the remaining collection of boxes, select those in $\mathcal{B}_2$ and discard those boxes in $\mathcal{B}_k, k \geq 3$, which lie inside these selected boxes, and so on. The lemma now follows from observations (i) and (ii) above. ■

By a **step function**, we mean a simple function as in (1.1), where the sets $E_i, 1 \leq i \leq k$, are all boxes.

2 Some approximation results

In this section, we prove some preliminary lemmas from which we will deduce the approximation results mentioned earlier.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $E \subset \Omega$ be a set of finite measure. Let $\varepsilon > 0$ be given. Then, there exists a set F, which is a finite disjoint union of boxes, such that*

$$\mu(E\Delta F) \ < \ \varepsilon,$$

where $E\Delta F = (E\backslash F) \cup (F\backslash E)$.

Proof: Since $\mu(E) < \infty$, there exists an open set $G \subset \mathbb{R}^N$ such that $E \subset G$ and $\mu(G\backslash E) < \frac{\varepsilon}{2}$. Set $G_0 = \Omega \cap G$. Then G_0 is also open, $E \subset G_0$ and $\mu(G_0\backslash E) < \frac{\varepsilon}{2}$. Now G_0 can be written as a countable disjoint union of boxes $\{I_j\}_{j=1}^{\infty}$ and we consequently have $I_j \subset \Omega$. Since $\sum_{j=1}^{\infty} \mu(I_j) < \infty$, let us choose k such that

$$\sum_{j=k+1}^{\infty} \mu(I_j) \ < \ \frac{\varepsilon}{2}.$$

Set $F = \cup_{j=1}^{k} I_j$, which is a finite disjoint union of boxes and we also have $F \subset G_0$. Now

$$\mu(F\backslash E) \ \leq \ \mu(G_0\backslash E) \ < \ \frac{\varepsilon}{2}$$

and

$$\mu(E\backslash F) \ \leq \ \mu(G_0\backslash F) \ \leq \ \sum_{j=k+1}^{\infty} \mu(I_j) \ < \ \frac{\varepsilon}{2}.$$

This completes the proof. ■

Lemma 2.2 *Let $E \subset \mathbb{R}^N$ be a set of finite measure. Let $\varepsilon > 0$ be given. Then there exists a compact set $K \subset E$ such that*

$$\mu(E\backslash K) \ < \ \varepsilon.$$

Proof: Step 1: Since E has finite measure, there exists an open set V such that $E \subset V$ and $\mu(V\backslash E) < \varepsilon$.

Let $B(0;r)$ denote the open ball centred at the origin and of radius r. The corresponding closed ball will be denoted $\overline{B}(0;r)$. For a positive integer n, set $V_n = B(0;n) \cap V$. Then $V_n \uparrow V$ and so there exists a positive integer m such that $\mu(V\backslash V_m) < \varepsilon$.Then

$$E\Delta V_m \ = \ (E\backslash V_m) \cup (V_m\backslash E) \ \subset \ (V\backslash V_m) \cup (V\backslash E)$$

and so $\mu(E\Delta V_m) < 2\varepsilon$.

Step 2: Now V_m is a bounded open set and so there exists $R > 0$ such that $V_m \subset B(0;R)$. Then, we can find $\rho > 0$ such that $\mu(B(0;R)\backslash\overline{B}(0;\rho)) < \varepsilon$. Further, there exists a closed set $F \subset V_m$ such that $\mu(V_m\backslash F) < \varepsilon$. Consequently, $\overline{B}(0;\rho) \cap F$ is compact and

$$\begin{aligned} V_m\backslash(\overline{B}(0;\rho) \cap F) &= (V_m\backslash\overline{B}(0;\rho)) \cup (V_m\backslash F) \\ &\subset (B(0;R)\backslash\overline{B}(0;\rho)) \cup (V_m\backslash F) \end{aligned}$$

and so $\mu(V_m\backslash(\overline{B}(0;\rho) \cap F)) < 2\varepsilon$.

Step 3: If $E \subset \mathbb{R}^N$ is of finite measure, we have seen in Step 1 above that there exists a bounded open set W such that $\mu(E\Delta W) < \frac{\varepsilon}{3}$. We have also seen in Step 2, that there exists a compact set $\widetilde{K} \subset W$ such that $\mu(W\backslash\widetilde{K}) < \frac{\varepsilon}{3}$. Finally, there exists a closed set $\widetilde{F} \subset E$ such that $\mu(E\backslash\widetilde{F}) < \frac{\varepsilon}{3}$. Then $K = \widetilde{F} \cap \widetilde{K}$ is compact, $K \subset E$ and

$$E\backslash K = (E\backslash W) \cup ((E \cap W)\backslash\widetilde{F}) \cup ((W \cap \widetilde{F})\backslash\widetilde{K})$$

from which it follows that $\mu(E\backslash K) < \varepsilon$. This completes the proof. ■

Lemma 2.3 *Let $I \subset \mathbb{R}^N$ be a box and let $\varepsilon > 0$ be given. Then there exists $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ such that $0 \leq \varphi(x) \leq 1$ for all x and*

$$\mu(\{x \in \mathbb{R}^N \mid \varphi(x) \neq \chi_I(x)\}) < \varepsilon.$$

Further, the support of φ will be contained in I.

Proof: Choose a closed box J_1 and an open box J_2 such that $J_1 \subset J_2 \subset \overline{J_2} \subset I$ and such that $\mu(I\backslash J_1) < \varepsilon$. By Urysohn's lemma, there exists a continuous function φ such that $0 \leq \varphi(x) \leq 1$ for all x and such that $\varphi(x) = 1$ for all $x \in J_1$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^N\backslash J_2$. Then the support of φ is contained in $\overline{J_2} \subset I$, which is compact and so $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$. Now,

$$\{x \in \mathbb{R}^N \mid \varphi(x) \neq \chi_I(x)\} \subset I\backslash J_1$$

and the result now follows immediately. ■

Remark 2.1 Urysohn's lemma is a result valid in the fairly general setting of a topological space which is normal (cf. Simmons [1]). It is much simpler in the context of a metric space and even simpler in $\mathbb{R}^N$. For example if we are in the real line and have a closed interval $[a, b]$ contained in an open interval (c, d), then we can easily construct a continuous function, which is identically equal to unity on $[a, b]$ and which vanishes outside (c, d), as follows:

$$\varphi(x) \; = \; \begin{cases} 0, & \text{if } x \leq c \text{ or } x \geq d, \\ \frac{x-c}{a-c}, & \text{if } c \leq x \leq a, \\ 1, & \text{if } a \leq x \leq b, \\ \frac{d-x}{d-b}, & \text{if} b \leq x \leq d. \end{cases}$$

A similar construction can be done in the case of boxes in $\mathbb{R}^N$. Another way to construct such functions (which is valid in any metric space (X, d)) is to define, for any set A, the distance of a point x from A by

$$d(x, A) \; = \; \inf_{y \in A} d(x, y);$$

this function is continuous. Now define

$$\varphi(x) \; = \; \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

If A and B are disjoint closed sets, then φ is a well-defined continuous function such that $\varphi \equiv 0$ on A and $\varphi \equiv 1$ on B. ■

Corollary 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let f be a step function defined on Ω and let $\varepsilon > 0$ be given. Then, there exists $\varphi \in \mathcal{C}_c(\Omega)$ such that*

$$\mu(\{x \in \Omega \mid \varphi(x) \neq f(x)\}) \; < \; \varepsilon \tag{2.1}$$

and such that

$$\|\varphi\|_\infty \; \leq \; \|f\|_\infty. \tag{2.2}$$

Proof: Let $f = \sum_{j=1}^k \alpha_i \chi_{I_j}$ where the I_j are all disjoint boxes contained in Ω. By Lemma 2.2 above, there exists $\varphi_j \in \mathcal{C}_c(\mathbb{R}^N)$ with support contained in $I_j \subset \Omega$, taking values in $[0, 1]$ and such that

$$\mu(\{x \in \mathbb{R}^N \mid \varphi_j(x) \neq \chi_{I_j}(x)\} \; < \; \frac{\varepsilon}{k}.$$

Set $\varphi = \sum_{j=1}^{k} \alpha_j \varphi_j$. Then

$$\begin{aligned}\{x \in \Omega \mid \varphi(x) \neq f(x)\} &\subset \cup_{j=1}^{k} \{x \in \Omega \mid \varphi_j(x) \neq \chi_{I_j}(x)\} \\ &\subset \cup_{j=1}^{k} \{x \in \mathbb{R}^N \mid \varphi_j(x) \neq \chi_{I_j}(x)\}\end{aligned}$$

and so

$$\mu(\{x \in \Omega \mid \varphi(x) \neq f(x)\}) < \varepsilon.$$

Since the supports of the φ_j are disjoint, it follows that

$$\|\varphi\|_\infty \le \max_{1 \le j \le k} |\alpha_j| = \|f\|_\infty.$$

Observe that the function φ has compact support contained in $\cup_{j=1}^{k} I_j \subset \Omega$. Thus $\varphi \in \mathcal{C}_c(\Omega)$. ■

3 Density of $\mathcal{C}_c(\Omega)$ in $L^p(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. Define

$$\mathcal{S} = \{\varphi : \Omega \to \mathbb{R} \mid \varphi \text{ is simple, and } \mu(\{x \in \Omega \mid \varphi(x) \neq 0\}) < \infty\}.$$

Observe that a simple function φ belongs to $L^p(\Omega)$, for $1 \le p < \infty$, if, and only if, it belongs to $\mathcal{S}$. Clearly, step functions are in $\mathcal{S}$.

Proposition 3.1 *Let $\mathcal{S}$ be defined as above. Then $\mathcal{S}$ is dense in $L^p(\Omega)$, for $1 \le p < \infty$.*

Proof: Let $f \in L^p(\Omega)$ be a non-negative function. Let $\{\varphi_n\}$ be simple functions such that $0 \le \varphi_n \le f$ and such that $\varphi_n \uparrow f$. Then, clearly $\varphi_n \in L^p(\Omega)$ and so $\varphi_n \in \mathcal{S}$, for each positive integer n. Since $|\varphi_n - f|^p \le 2^p |f|^p$, and since $|f|^p$ is integrable, it follows from the dominated convergence theorem that $\varphi_n \to f$ in $L^p(\Omega)$. If f is real valued, the result can be deduced by observing that $f = f^+ - f^-$ where $f^+ = \max\{f, 0\} \ge 0$ and $f^- = -\min\{f, 0\} \ge 0$. If f is complex valued, we can apply the preceding arguments to $\mathrm{Re} f$ and $\mathrm{Im} f$ to obtain the same conclusion for f. This completes the proof. ■

Proposition 3.2 *Let $1 \le p < \infty$. Every function $f \in \mathcal{S}$ can be approximated in $L^p(\Omega)$ by step functions.*

Proof: Let $f = \chi_E$ where $E \subset \Omega$ is a set of finite measure. Let $\varepsilon > 0$. Then, by Lemma 2.1, there exists a set F, which is a finite disjoint union of boxes, such that $\mu(E\Delta F) < \varepsilon^p$. Then

$$\|\chi_E - \chi_F\|_p^p \;=\; \mu(E\Delta F) \;<\; \varepsilon^p$$

and so $\|\chi_E - \chi_F\|_p < \varepsilon$. Now, let $f = \sum_{j=1}^k \alpha_j \chi_{E_j}$, where the E_j are mutually disjoint sets of finite measure and all the α_j are non-zero. For each $1 \leq j \leq k$, choose F_j, a finite disjoint union of boxes such that $\|\chi_{E_j} - \chi_{F_j}\|_p < \frac{\varepsilon}{k|\alpha_j|}$. Consider the step function $\varphi = \sum_{j=1}^k \alpha_j \chi_{F_j}$. Then, by the triangle inequality, it follows that $\|f - \varphi\|_p < \varepsilon$. This completes the proof. ■

Proposition 3.3 *Every step function defined on* Ω *can be approximated by a function from* $\mathcal{C}_c(\Omega)$ *in* $L^p(\Omega)$*, where* $1 \leq p < \infty$*.*

Proof: Let $f \neq 0$ be a step function defined on Ω. Let $\varepsilon > 0$. By Corollary 2.1, there exists $\varphi \in \mathcal{C}_c(\Omega)$ such that

$$\mu(\{x \in \Omega \mid \varphi(x) \neq f(x)\}) \;<\; \left(\frac{\varepsilon}{2\|f\|_\infty}\right)^p$$

and such that (2.2) holds. Then

$$\|\varphi - f\|_p^p \;\leq\; 2^p \|f\|_\infty^p \mu(\{x \in \Omega \mid \varphi(x) \neq \chi_I(x)\}) \;<\; \varepsilon^p.$$

Thus $\|\varphi - f\|_p < \varepsilon$. This completes the proof. ■

Combining Propositions 3.1-3.3, we deduce the following result.

Theorem 3.1 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $1 \leq p < \infty$*. Then* $\mathcal{C}_c(\Omega)$ *is dense in* $L^p(\Omega)$*.* ■

4 Lusin's theorem

Theorem 4.1 (Lusin) *Let* $E \subset \mathbb{R}^N$ *be a measurable set of finite measure. Let* $f : E \to \mathbb{R}$ *be a measurable function. Let* $\varepsilon > 0$ *be given. Then, there exists* $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ *such that*

$$\mu(\{x \in E \mid \varphi(x) \neq f(x)\}) \;<\; \varepsilon.$$

Further, if f *is bounded then we can ensure that*

$$\|\varphi\|_\infty \;\leq\; \|f\|_\infty.$$

Proof: For each positive integer n, define

$$E_n = \{x \in E \mid |f(x)| \leq n\}.$$

Then $E_n \uparrow E$. Choose N such that $\mu(E \backslash E_N) < \frac{\varepsilon}{3}$. Now define $\widetilde{f} : \mathbb{R}^N \to \mathbb{R}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E_N, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash E_N. \end{cases}$$

Since $\widetilde{f}$ is bounded and since E_N has finite measure, it follows that $\widetilde{f}$ is integrable on $\mathbb{R}^N$. Consequently, by Theorem 3.1, there exists a sequence $\{\varphi_n\}$ in $\mathcal{C}_c(\mathbb{R}^N)$ such that $\varphi_n \to \widetilde{f}$ in $L^1(\mathbb{R}^N)$. Then, there exists a subsequence $\{\varphi_{n_k}\}$ which converges to $\widetilde{f}$ almost everywhere.

Now, since E_N has finite measure, there exists $F \subset E_N$ such that $\mu(E_N \backslash F) < \frac{\varepsilon}{3}$ and such that $\varphi_{n_k} \to \widetilde{f}$ uniformly on F, by virtue of Egorov's theorem. Again, since F has finite measure, we can choose a compact set $K \subset F$ such that $\mu(F \backslash K) < \frac{\varepsilon}{3}$ (cf. Lemma 2.2). Clearly $\mu(E \backslash K) < \varepsilon$.

Now, since φ_{n_k} converges uniformly to $\widetilde{f}$ on K, it follows that the restriction of $\widetilde{f}$ to K is continuous on K. But $K \subset F \subset E_N$ and so $\widetilde{f} = f$ on K. Thus, the restriction of f to K is continuous on K. We can then extend the restriction of f to K, to all of $\mathbb{R}^N$ by means of the Tietze extension theorem. If f were bounded, this function, denoted g, will also satisfy

$$\|g\|_\infty \leq \|f\|_\infty.$$

Finally, let $\psi \in \mathcal{C}_c(\mathbb{R}^N)$ be such that $0 \leq \psi \leq 1$ and such that $\psi \equiv 1$ on K and set $\varphi = \psi g$. Then $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$ and

$$\{x \in E \mid \varphi(x) \neq f(x)\} \subset E \backslash K$$

and $\mu(E \backslash K) < \varepsilon$. This completes the proof. ■

Remark 4.1 The extension theorem of Tietze is equivalent to Urysohn's lemma in a normal topological space (cf. Simmons [1]). ■

References

[1] Simmons, G. F. Introduction to Topology and Modern Analysis, McGraw-Hill, 1963.