Korovkin's Theorem - Revisited

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Abstract

Korovkin's theorem is an abstract result in approximation theory which gives conditions for uniform approximation of continuous functions on a compact metric space using sequences of positive linear operators (on the space of continuous functions). It gives simple proofs of some major approximation theorems in analysis like the Weierstrass approximation theorem and Fejér's theorem for the Cesàro summability of Fourier series. We state and prove a measure theoretic version of Korovkin's theorem which, to the best of our knowledge, seems new and which is very simple to establish. We also show how the theorems mentioned above can be deduced from this.

1 Introduction

The theorem of Korovkin is an abstract result which uses sequences of positive linear operators on the space of continuous real valued functions on a compact metric space to provide uniform approximations of continuous functions. It is stated as follows.

Theorem 1.1 *(*Korovkin *[5]) Let (X, d) be a compact metric space and let $\mathcal{C}(X)$ denote the space of continuous real valued functions on X with the sup-norm. Let $A_n : \mathcal{C}(X) \to \mathcal{C}(X)$ be a sequence of linear maps which are positive (i.e. if $f \in \mathcal{C}(X)$ is such that $f(x) \geq 0$ for all $x \in X$, then $(A_n f)(x) \geq 0$ for all $x \in X$). Consider the following hypotheses:*
(H1) *If $f_0(x) = 1$ for all $x \in X$, then*

$$\lim_{n\to\infty} \|A_n f_0 - f_0\| \; = \; 0.$$

(H2) *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\varphi(t) > 0$ for $t > 0$ and, for $x \in X$, let*

$$\psi_x(y) \;\stackrel{\text{def}}{=}\; \varphi(d(x, y)), \; y \in X.$$

Then

$$\lim_{n\to\infty} \left(\sup_{x\in X} |(A_n\psi_x)(x)| \right) \; = \; 0.$$

If (H1) and (H2) hold, then for every $f \in \mathcal{C}(X)$, we have

$$\lim_{n\to\infty} \|A_n f - f\| \; = \; 0. \; \blacksquare$$

A proof of this result can be found in Ciarlet [3]. Two important consequences of this result are known as Korovkin's first theorem (also known as the Korovkin-Bohman theorem or just as Bohman's theorem - cf. Bohman [2]) and Korovkin's second theorem.

Theorem 1.2 *(Korovkin's first theorem) Let $A_n : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be a sequence of positive linear maps such that*

$$\lim_{n\to\infty} \|A_n f_i - f_i\| \;=\; 0 \text{ for } i = 0, 1, 2,$$

where $f_i(t) = t^i$ for $i = 0, 1, 2$. Then for every $f \in \mathcal{C}[0,1]$ we have

$$\lim_{n\to\infty} \|A_n f - f\| \;=\; 0. \blacksquare$$

Theorem 1.3 *(Korovkin's second theorem) Let $\mathcal{C}_{\text{per}}[-\pi,\pi]$ denote the space of all continuous 2π-periodic real valued functions defined on the interval $[-\pi,\pi]$. Let $A_n : \mathcal{C}_{\text{per}}[-\pi,\pi] \to \mathcal{C}_{\text{per}}[-\pi,\pi]$ be a sequence of positive linear maps such that*

$$\lim_{n\to\infty} \|A_n g_i - g_i\| \;=\; 0 \text{ for } i = 0, 1, 2,$$

where, for all $t \in [-\pi,\pi]$,

$$g_0(t) \;=\; 1,\;\; g_1(t) \;=\; \cos t \text{ and } g_2(t) \;=\; \sin t.$$

Then for every $f \in \mathcal{C}_{\text{per}}[-\pi,\pi]$, we have

$$\lim_{n\to\infty} \|A_n f - f\| \;=\; 0. \blacksquare$$

A beautiful application of Korovkin's first theorem is a simple proof of the Weierstrass approximation theorem wherein a continuous function on $[0,1]$ is uniformly approximated by a sequence of Bernstein polynomials. An application of the second theorem of Korovkin is a proof of Fejér's theorem which establishes the uniform Cesàro summability of the Fourier series of a continuous 2π-periodic function on $[-\pi,\pi]$.

For a proof of all these results see Altomare [1] or Ciarlet [3].

In this article, we will present a measure theoretic version of Korovkin's theorem and derive the Weierstrass and Fejér theorems from it. We will also show that the original theorem of Korovkin follows from it. This version is very easy to prove and is a nice example of the technique of 'divide and rule' which is very useful in estimating integrals: we split an integral over two complementary sets such that on one of them we control the integrand, while the other set, where we have no control, is of small measure.

Throughout the sequel the symbol $\mathbb{N}$ will stand for the set of natural numbers $\{1, 2, 3, \cdots\}$.

2 Korovkin's theorem

Theorem 2.1 *Let (X, d) be a compact metric space. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function such that $\varphi(t) > 0$ for $t > 0$. Let $\{\mu_n^x\}_{x \in X, n \in \mathbb{N}}$ be a collection of finite (Borel) measures on X. Define:*

$$\psi_n(x) \;=\; \int_X \varphi(d(x, y))\; d\mu_n^x(y) \text{ for } x \in X.$$

Assume that the following hypotheses hold:
(H3) $\mu_n^x(X) \to 1$ uniformly on X as $n \to \infty$;
(H4) $\psi_n(x) \to 0$ uniformly on X as $n \to \infty$.
Then for any $f \in \mathcal{C}(X)$, we have that

$$\int_X f(y)\; d\mu_n^x(y) \;\to\; f(x)$$

uniformly on X as $n \to \infty$.

Proof: Since X is compact, $\sup_{x,y \in X} d(x, y)$ is finite and is attained. Let

$$\operatorname{diam}(X) \;=\; \max_{x,y \in X} d(x, y).$$

For $0 < \delta < \operatorname{diam}(X)$, let

$$\alpha(\delta) \;\stackrel{\text{def}}{=}\; \min_{t \in [\delta, \operatorname{diam}(X)]} \varphi(t).$$

Notice that $\alpha(\delta) > 0$ for $\delta > 0$. Given $\delta > 0$ and $x \in X$, define

$$A_\delta(x) \;=\; \{y \in X \mid d(x, y) \geq \delta\}.$$

Then, for any $x \in X$ and for any $n \in \mathbb{N}$, we have

$$\alpha(\delta)\mu_n^x(A_\delta(x)) \;\leq\; \int_{A_\delta(x)} \varphi(d(x, y))\; d\mu_n^x(y) \;\leq\; \psi_n(x). \tag{2.1}$$

Since $f \in \mathcal{C}(X)$ is uniformly continuous, given $\eta > 0$, we can find $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $d(x, y) < \delta$.

Set

$$\Delta_n(x) \;=\; \left| \int_X f(y)\; d\mu_n^x(y) - f(x) \right|.$$

Now

$$\int_X f(y)\; d\mu_n^x(y) - f(x) \;=\; \int_X (f(y) - f(x))\; d\mu_n^x(y) + f(x)(\mu_n^x(X) - 1). \tag{2.2}$$

To estimate this, we split the integral $\int_X |f(y) - f(x)|\; d\mu_n^x(y)$ into two parts: one over $A_\delta(x)$ and the other over its complement. By the choice of δ, we have

$$\int_{X \setminus A_\delta(x)} |f(y) - f(x)|\; d\mu_n^x(y) \;<\; \eta\mu_n^x(X \setminus A_\delta(x)) \;\leq\; \eta\mu_n^x(X) \;=\; \eta + \eta(\mu_n^x(X) - 1). \tag{2.3}$$

On the other hand, using (2.1), we get

$$\int_{A_\delta(x)} |f(y) - f(x)| \, d\mu_n^x(y) \;\leq\; 2\|f\| \mu_n^x(A_\delta(x)) \;\leq\; \frac{2\|f\|}{\alpha(\delta)} \psi_n(x), \tag{2.4}$$

where $\|f\|$ stands for the usual sup-norm of f. Combining (2.2), (2.3) and (2.4), we get

$$\Delta_n(x) \;\leq\; \eta + \frac{2\|f\|}{\alpha(\delta)} \psi_n(x) + (\eta + \|f\|)|\mu_n^x(X) - 1|.$$

Now, given $\varepsilon > 0$, first choose $\eta < \frac{\varepsilon}{3}$ and then fix δ using the uniform continuity of f. Now, using hypotheses (H3) and (H4), we can find $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in X$, we have

$$\begin{aligned} (\eta + \|f\|)|\mu_n^x(X) - 1| &< \tfrac{\varepsilon}{3}, \\ \tfrac{2\|f\|}{\alpha(\delta)} \psi_n(x) &< \tfrac{\varepsilon}{3}, \end{aligned}$$

which yields

$$\Delta_n(x) \;<\; \varepsilon \text{ for all } x \in X, n \geq N.$$

This completes the proof. ∎

We can deduce the original version of Korovkin's theorem (Theorem 1.1) from the above version. Indeed, let $\{A_n\}$ be a sequence of positive linear maps from $\mathcal{C}(X)$ into itself. For each $x \in X$, the map $f \mapsto (A_n f)(x)$ defines a positive linear functional on $\mathcal{C}(X)$. Then, by the Riesz representation theorem (cf. Rudin [7]), there exists a finite Borel measure μ_n^x on X such that

$$(A_n f)(x) \;=\; \int_X f(y) \, d\mu_n^x(y).$$

Now it is clear to see that the hypotheses (H1) and (H2) are exactly the same as hypotheses (H3) and (H4) respectively. Thus Theorem 1.1 is the same as Theorem 2.1 in this context.

3 Weierstrass' approximation theorem

Let $X = [0, 1]$. Let δ_x denote the Dirac measure concentrated at $x \in X$, *i.e.* if $E \subset X$, then

$$\delta_x(E) \;=\; \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Define, for $n \in \mathbb{N}$ and for $t \in [0, 1]$,

$$\mu_n^t \;=\; \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \delta_{\frac{k}{n}}.$$

Then

$$\mu_n^t(X) \;=\; \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \;=\; (t + 1 - t)^n \;=\; 1.$$

Thus (H3) is trivially satisfied. Now

$$\begin{aligned}\int_X s\ d\mu_n^t(s) &= \textstyle\sum_{k=0}^n \frac{k}{n} \binom{n}{k} t^k(1-t)^{n-k} \\ &= \textstyle\sum_{k=1}^n \binom{n-1}{k-1} t^k(1-t)^{n-k} \\ &= t.\end{aligned}$$

In the same way (exercise!) we can see that

$$\int_X s^2\ d\mu_n^t(s) = \frac{1}{n}((n-1)t^2+t).$$

Now let us choose $\varphi(t) = t^2$. Then

$$\begin{aligned}\psi_n(t) &= \textstyle\int_X (t-s)^2\ d\mu_n^t(s) \\ &= t^2\mu_n^t(X) - 2t\textstyle\int_X s\ d\mu_n^t(s) + \int_X s^2\ d\mu_n^t(s) \\ &= \tfrac{t-t^2}{n}.\end{aligned}$$

Since, for all $t \in [0,1]$, we have that $t - t^2 \le \frac{1}{4}$, it follows that

$$\psi_n(t) \le \frac{1}{4n}$$

for all $t \in [0,1]$ which establishes the validity of the hypothesis (H4). Thus, by Theorem 2.1, we have that for all $f \in \mathcal{C}[0,1]$,

$$\int_X f(s)\ d\mu_n^t(s) \to f(t)$$

uniformly on $[0,1]$ as $n \to \infty$. In other words,

$$B_n(t) \stackrel{\text{def}}{=} \sum_{k=0}^n f\left(\frac{k}{n}\right)\binom{n}{k} t^k(1-t)^{n-k} \to f(t)$$

uniformly on $[0,1]$ as $n \to \infty$. The functions $B_n(t)$ are called the **Bernstein polynomials**. This proves the Weierstrass approximation theorem which states that any continuous function on $[0,1]$ (or, for that matter, any finite interval $[a,b]$) can be uniformly approximated in that interval by a sequence of polynomials.

Probabilistic proofs of Weierstrass' theorem also proceed on similar lines (cf. Parzen [6]).

We can easily prove the following (equivalent) version of Bohman's theorem (Theorem 1.2) using the same kind of arguments as in this section.

Theorem 3.1 *Let* $\{\mu_n^t\}_{t\in[0,1],n\in\mathbb{N}}$ *be a collection of finite (Borel) measures on* $[0,1]$ *such that*

$$\int_{[0,1]} s^i \; d\mu_n^t(s) \;\to\; t^i$$

uniformly on $[0,1]$ *as* $n \to \infty$ *for* $i = 0, 1, 2$*. Then for any* $f \in \mathcal{C}[0,1]$*, we have that*

$$\int_{[0,1]} f(s) \; d\mu_n^t(s) \;\to\; f(t)$$

uniformly on $[0,1]$ *as* $n \to \infty$*.*

Proof: Exercise ! ■

4 Fejér's theorem

Let us denote by $\mathcal{C}_{\text{per}}[-\pi,\pi]$ the space of continuous 2π periodic real valued functions defined on the interval $[-\pi,\pi]$. If $f \in \mathcal{C}_{\text{per}}[-\pi,\pi]$, its Fourier series is given by

$$f(x) \;\sim\; \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$\begin{aligned} a_k &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(t)\cos kt \; dt, \quad \text{for } k \in \{0\}\cup\mathbb{N}, \\ b_k &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(t)\sin kt \; dt, \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{4.5}$$

For negative integers k, we set $a_k = a_{-k}, b_k = -b_{-k}$ and also set $b_0 = 0$. Let $\{s_n(x)\}$ denote the sequence of partial sums of the Fourier series for f:

$$s_n(x) \;=\; \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx).$$

If we set

$$c_k \;=\; \frac{1}{2}(a_k - ib_k) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt} \; dt,$$

where $i^2 = -1$, we get

$$s_n(x) \;=\; \sum_{k=-n}^{n} c_k e^{ikx} \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t) \; dt$$

where

$$D_n(t) \;\stackrel{\text{def}}{=}\; \sum_{k=-n}^{n} e^{ikt}.$$

This is called the **Dirichlet kernel**. Since f and D_n are both 2π-periodic, we get, by a simple change of variable, that

$$s_n(x) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t) \; dt \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t) \; dt.$$

Lemma 4.1 *Let $n \geq 0$ be a non-negative integer. Then*

$$D_n(t) \;=\; \begin{cases} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} & \text{if } t \neq 2m\pi, m \in \mathbb{N} \cup \{0\}, \\ 2n+1 & \text{if } t = 2m\pi, m \in \mathbb{N} \cup \{0\}. \end{cases} \tag{4.6}$$

Further

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\; dt \;=\; 1. \tag{4.7}$$

Proof: If $t = 2m\pi$ for $m \in \mathbb{N} \cup \{0\}$, then, $D_n(t) = 2n+1$ by definition. Let us now assume that $t \neq 2m\pi, m \in \mathbb{N} \cup \{0\}$. We have

$$(e^{it} - 1)D_n(t) \;=\; e^{i(n+1)t} - e^{-int}.$$

Multiplying both sides of this relation by $e^{-i\frac{t}{2}}$, we immediately deduce (4.6). The relation (4.7) follows from the following:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\; dt \;=\; \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if} k \neq 0. \;\blacksquare \end{cases}$$

Now let us define the **Fejér kernel** for a non-negative integer n:

$$K_n(t) \;\stackrel{\text{def}}{=}\; \frac{1}{n+1}\sum_{k=0}^{n} D_k(t).$$

Lemma 4.2 *Let $n \geq 0$ be a non-negative integer. Then the following statements are valid.*
(i) *If $t \neq 2m\pi, m \in \mathbb{N} \cup \{0\}$,*

$$K_n(t) \;=\; \frac{1}{n+1}\frac{1-\cos(n+1)t}{1-\cos t} \;=\; \frac{1}{n+1}\frac{\sin^2\frac{(n+1)t}{2}}{\sin^2\frac{t}{2}} \;\geq\; 0. \tag{4.8}$$

(ii)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\; dt \;=\; 1. \tag{4.9}$$

(iii) *If $0 < \delta \leq |t| \leq \pi$, then*

$$0 \;\leq\; K_n(t) \;\leq\; \frac{1}{(n+1)\sin^2\frac{\delta}{2}}. \tag{4.10}$$

Proof: Notice that (cf. proof of Lemma 4.1)

$$\begin{aligned} (e^{-it} - 1)(e^{it} - 1)(n+1)K_n(t) \;&=\; (e^{-it} - 1)\textstyle\sum_{k=0}^{n}(e^{i(k+1)t} - e^{-ikt}) \\ &=\; 2 - e^{i(n+1)t} - e^{-i(n+1)t}, \end{aligned}$$

from which we immediately deduce (4.8). The relation (4.9) follows directly from the definition of $K_n(t)$ and the relation (4.7). Relation (4.10) is a direct consequence of (4.8). $\blacksquare$

Given $f \in \mathcal{C}_{\text{per}}[-\pi, \pi]$, let us set

$$\sigma_n(x) \;=\; \frac{1}{n+1}(s_0(x) + s_1(x) + \cdots + s_n(x)).$$

It then follows from the expression for the partial sums of the Fourier series $s_n(x)$ that

$$\sigma_n(x) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)\;dt \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\;dt. \qquad (4.11)$$

The last equality again follows from the fact that f is 2π-periodic, by a simple change of variable.

Based on an analysis of the Dirichlet kernel, a nice application of the Banach-Steinhaus theorem (uniform boundedness principle) in functional analysis shows that there exists a continuous 2π-periodic function on $[-\pi, \pi]$ (in fact, a dense G_δ set of functions in $\mathcal{C}_{\text{per}}[-\pi, \pi]$) whose Fourier series diverges at the point $x = 0$. The same argument can be used to show the divergence of the Fourier series at any point in $(-\pi, \pi)$ for a large set of functions. Combining this argument with the Baire category theorem, we can, in fact, show that there exists a dense G_δ set of functions in $\mathcal{C}_{\text{per}}[-\pi, \pi]$ for each of which the Fourier series will diverge on a dense G_δ set of points in $(-\pi, \pi)$. In a metric space with no isolated points, a dense G_δ set has to be uncountable. Thus, we have that there exist uncountably many continuous 2π-periodic functions on $[-\pi, \pi]$ such that for each one of them, the Fourier series diverges at an uncountably many number of points in $[-\pi, \pi]$. The reader is referred to Kesavan [4] or to Rudin [7] for details.

Thus, without further hypotheses on a continuous 2π-periodic function, one cannot expect its Fourier series to converge to the value of the function at any point. However, we have the following result of uniform convergence of a sequence of trigonometric polynomials to a continuous 2π-periodic function. The existence of such an approximation is guaranteed by the Stone-Weierstrass theorem.

Theorem 4.1 *(Fejér) Let $f \in \mathcal{C}_{\text{per}}[-\pi, \pi]$. With the notations established earlier, we have that $\sigma_n \to f$ uniformly on $[-\pi, \pi]$.*

Proof: We will give a proof based on Korovkin's theorem. Notice that, in view of (4.11), we need to show that for any 2π-periodic function f defined on $[-\pi, \pi]$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)K_n(\theta - \tau)\;d\tau$$

converges uniformly to $f(\theta)$ as θ varies over $[-\pi, \pi]$.

Consider the compact metric space S^1, the unit circle in the plane with centre at the origin, equipped with the usual euclidean distance metric. Then, any point $x \in S^1$ can be written as $x = (\cos\theta, \sin\theta)$, where $\theta \in [-\pi, \pi]$. It is then clear that $\mathcal{C}(S^1)$ is in bijective correspondence with $\mathcal{C}_{\text{per}}[-\pi, \pi]$ via the mapping $f \mapsto \widetilde{f}$, where

$$\widetilde{f}(\theta) \;=\; f(x),\;\; x = (\cos\theta, \sin\theta).$$

Let us now define the measure μ_n^x on S^1, where $n \in \mathbb{N}$ and $x = (\cos\theta, \sin\theta) \in S^1$ such that the relation

$$\int_{S^1} f(y)\ d\mu_n^x(y) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{f}(\tau)K_n(\theta-\tau)\ d\tau \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{f}(\theta-\tau)K_n(\tau)\ d\tau$$

is valid for any $f \in \mathcal{C}(S^1)$. Then, for any $x \in S^1$, relation (4.9) gives

$$\mu_n^x(S^1) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\tau)\ d\tau \;=\; 1.$$

Thus hypothesis (H3) is trivially satisfied.

Once again, let us take $\varphi(t) = t^2$. Then, if $x = (\cos\theta, \sin\theta)$ and $y = (\cos\tau, \sin\tau)$ are points in S^1, elementary geometry shows that

$$|x-y|^2 \;=\; 4\sin^2\left(\frac{\theta-\tau}{2}\right).$$

Now

$$\begin{aligned} 4\sin^2\left(\tfrac{\theta-\tau}{2}\right) &= 2-2\cos(\theta-\tau) \\ &= 2-2\cos\theta\cos\tau-2\sin\theta\sin\tau. \end{aligned}$$

Thus

$$\begin{aligned} \psi_n(x) &= \textstyle\int_X |x-y|^2\ d\mu_n^x(y) \\ &= \tfrac{2}{2\pi}\textstyle\int_{-\pi}^{\pi} K_n(\theta-\tau)\ d\tau - \tfrac{2\cos\theta}{2\pi}\textstyle\int_{-\pi}^{\pi}\cos\tau K_n(\theta-\tau)\ d\tau \\ &\quad -\tfrac{2\sin\theta}{2\pi}\textstyle\int_{-\pi}^{\pi}\sin\tau K_n(\theta-\tau)\ d\tau. \end{aligned}$$

By vitrue of (4.9), the first term on the right-hand side is equal to 2. Now notice that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\tau K_n(\theta-\tau)\ d\tau$$

is nothing but $\sigma_n(\theta)$ when $f(\theta) = \cos(\theta)$. But the Fourier series of the cosine function consists of just one term *viz.* $\cos\theta$! Consequently when f is the cosine function, we have that

$$\sigma_n(\theta) \;=\; \frac{n}{n+1}\cos\theta.$$

Hence the second term on the right-hand side is equal to

$$\frac{2n}{n+1}\cos^2\theta.$$

By an identical reasoning, the third term reduces to

$$\frac{2n}{n+1}\sin^2\theta.$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, we finally get

$$\psi_n(x) \;=\; 2\left(1-\frac{n}{n+1}\right)(\cos^2\theta+\sin^2\theta) \;=\; \frac{2}{n+1}.$$

Thus ψ_n converges uniformly on S^1 to zero and so we can now apply Korovkin's theorem (Theorem 2.1) to conclude the proof. ■

Remark 4.1 The proof of Korovkin's second theorem (Theorem 1.3) follows the same lines as the above proof (cf. Ciarlet [3]). ■

Remark 4.2 We do not really need Korovkin's theorem to prove Fejér's theorem. If $f \in \mathcal{C}_{\text{per}}[-\pi, \pi]$, then using (4.11) and (4.9), we can write

$$\sigma_n(x) - f(x) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t) - f(x))K_n(t)\; dt.$$

Now we can easily estimate this integral using the 'divide and rule' principle and statement (iii) of Lemma 4.2 to deduce the uniform convergence (exercise!). ■

Remark 4.3 A series is said to be **Cesàro summable** if the sequence of averages of the partial sums converges. Thus the Fourier series of $f \in \mathcal{C}_{\text{per}}[-\pi, \pi]$ is uniformly Cesàro summable to f over $[-\pi, \pi]$. ■

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