Geometry and Symmetry
The work of Gromov

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The Institute of Mathematical Sciences

5 April 2009

Research in Geometry?

Conversation in a train:

Co-passenger What are you scribbling on that notebook?

Geometer I do research in Geometry.

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Some Important Geometric Figures

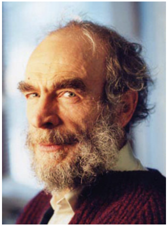
René Descartes

Bernhard Riemann

Sophus Lie

Felix Klein

Élie Cartan



Mikhail Gromov

A Group of Axioms

A Group is a set G, with a special element e called the *identity* element, a binary operation μ called *group multiplication* and a unary operation ι called *inversion* such that:

- The operation μ is *associative*

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$$

- The element e is a multiplicative identity

$$\mu(a, e) = e = \mu(e, a)$$

- The inversion of an element provides a multiplicative inverse

$$\mu(a, \iota(a)) = e = \mu(\iota(a), a)$$

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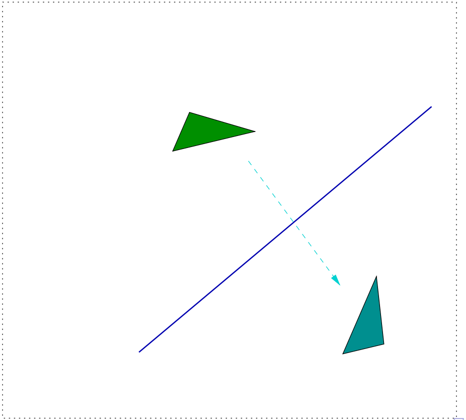
Reflections on Geometry

High school (or Euclidean) geometry can be seen as the properties of figures that are preserved under the group generated by reflections. This group also contains rotations and translations.



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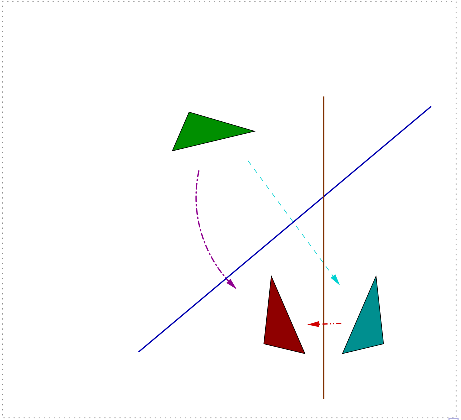

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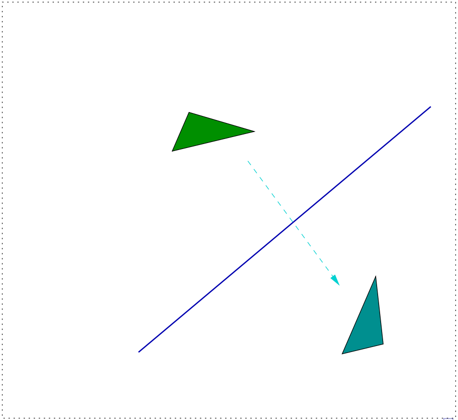

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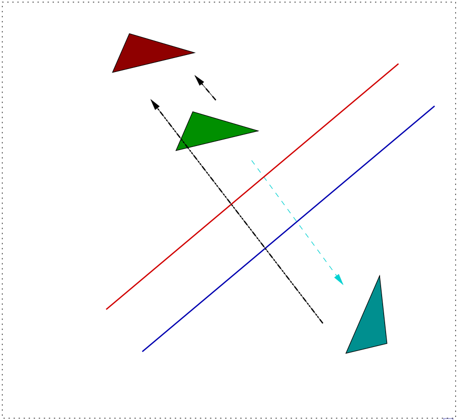

The Euclidean Group

The group generated by reflections can be described in co-ordinate terms as the collection of all transformations of the form

$$p \mapsto Ap + \mathbf{w}$$

where,

- p denotes a point of the plane written as a vertical vector.
- A is a 2×2 orthogonal matrix ($AA^t = \mathbf{1}$).
- $\mathbf{w}$ is some vector.

This is an example of a Lie (or continuous or topological) group. The group operations are continuous.

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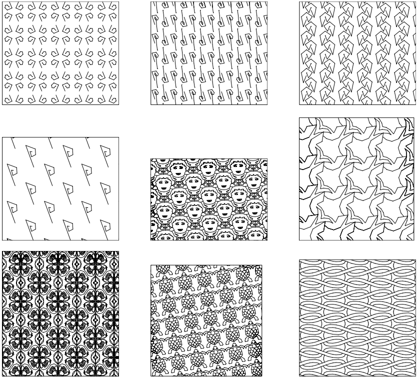
Discrete Groups enter quietly

There are also *discrete* groups of plane transformations that create interesting patterns:



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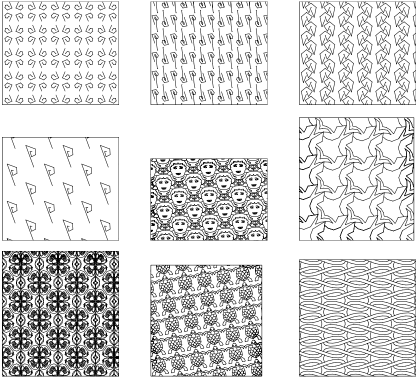

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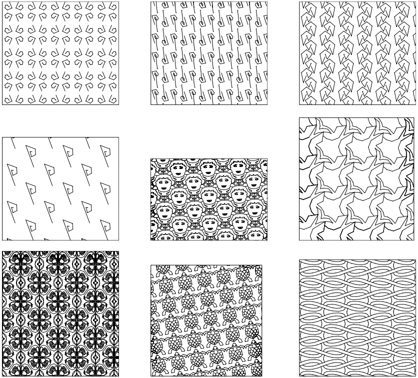

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The Poincaré disk

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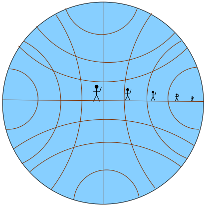

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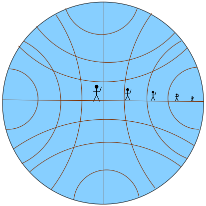

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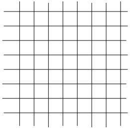

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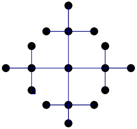

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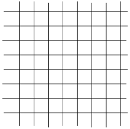

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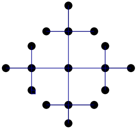

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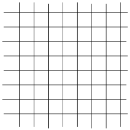

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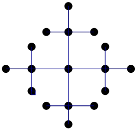

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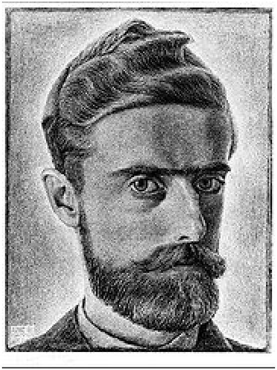

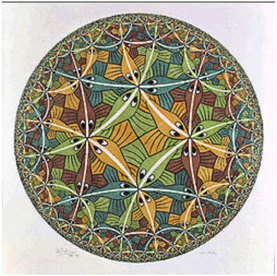

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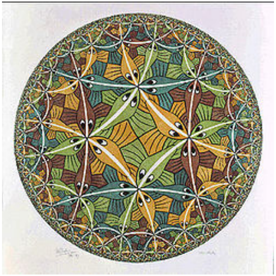

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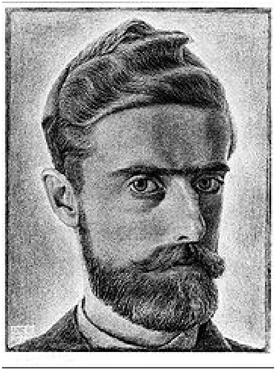

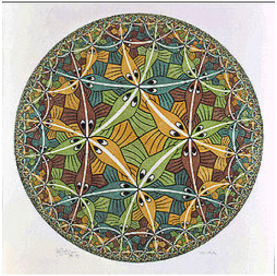

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Erlangen Programme

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Riemann was the man

Bernhard Riemann had taken a different approach.

> *the fundamental geometric notion is that of the measurement of "speed".*

Given a path between two points the length of the path is determined by integrating the speed.

The geodesic is the "path of least resistance" or "energy minimising path". It plays the role of "straight line" in the geometry of this space.

Thus Riemann could construct and study geometries that were *not* homogeneous.

In some sense in-homogeneity is closer to physical reality, with homogeneity being a Platonic ideal.

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Cartan blanched

Élie Cartan was able to find a way to combine the work of Riemann on the one hand and the work of Lie and Klein on the other. His fundamental idea is:

the infinitesimal *geometry at any two points is isomorphic*

The theory of invariants tensors allows us to make this precise. Thus infinitesimal geometry is understood using the geometries of Klein and Lie, while allowing the *development* (or integral) of these structures to evolve in-homogeneously.

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The work of Cartan provides the framework to study differential geometry even today. So it looked as if the final word on what is a geometry had been written!

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A spider's web

The kind of tiling (a discrete structure) that is available for a homogeneous geometry closely reflects the underlying (continuous) geometry.

For example, the tiling groups for the Euclidean plane contain the lattice which is a free *abelian* group, whereas the tiling groups for the Poincaré disk contain the free (non-commutative) group.

The free group $F(a, b)$ on two generators a and b consists of all words in the symbols a, b, a^{-1} and b^{-1}; where we are allowed to add and drop the pairs

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Large scale structure

Mikhail Gromov was able to define a precise relation-ship between the discrete and the continuous via the notion of quasi-isometry.
A (set-theoretic) map $f : X \to Y$ is said to be a (k, c) *quasi-isometry* if for any pair of points u, v in X we have

$$\begin{aligned} d(u,v) &\leq kd(f(u),f(v))+c \\ d(f(u),f(v)) &\leq kd(u,v)+c \end{aligned}$$

This is a way of “tearing” and “stitching together” that leaves invariant certain kinds of “hyperbolicity” of the space.

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Hyperbolicity

The fundamental property of non-Euclidean space like the Poincaré disk is that "geodesic triangles are thin".

> *If o is a point of the space and oa and ob are two distinct geodesics through o, then the geodesic from a to b goes* close *to o.*

If you go far enough from o in different directions, then the distance between the points on these two geodesis grows exponentially in terms of the distance from o.
One can show that this property carries over to higher dimensional fugures.

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Almost-, Quasi-, Pseudo-

The other big advantange of this approach is that it allows us to study space that are quasi-isometric to homogeneous spaces without themselves being homogeneous.This allows us to carry over results from Group theory to Geometry and vice versa.
So one can say that under Gromov's theory, group theory and geometry have been unified.