▸ KAMAL LODAYA AND A.V. SREEJITH, *Counting quantifiers and linear arithmetic on word models.*

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It is well known that first order logic cannot express various kinds of counting. One option is to add counting quantifiers, but there are many ways of doing this, see for example [8, 1, 14, 11]. Let us introduce some specific syntax for word models: we use a set of variables Var and in the signature we have binary predicates $<$ and Suc (the latter is definable using the former but may not be in fragments of the logic) and two constants min, max for the ordering, and unary predicates $a,\ a \in \Sigma$, for the finite alphabet Σ.

$$FO \quad \begin{array}{l} t ::= c,\ c \in \mathbb{N} \mid min \mid max \mid x,\ x \in Var \\ \alpha ::= a(t),\ a \in \Sigma \mid t_1 < t_2 \mid Suc(t_1, t_2) \mid t_1 = t_2 \mid \kappa \mid \neg\, \alpha \mid \alpha\ \vee\ \beta \mid \exists x \alpha \end{array}$$

We now introduce the syntax for the counting capabilities κ which extend FO. In the most general setting FOunC, we have counting terms and allow their comparison. This logic can define addition [5]. Hence, by an old result of Robinson [10], in the presence of the unary predicates (for $|\Sigma| \geq 2$), satisfiability is undecidable.

$$FO\text{unC} \quad \begin{array}{l} t ::= c,\ c \in \mathbb{N} \mid min \mid max \mid x,\ x \in Var \mid \#x\alpha \\ \kappa ::= t_1 \sim t_2, \sim \in \{<,=\} \mid t \equiv r(\mod q) \end{array}$$

In the more restricted FOmod, counting terms cannot be compared with each other:

$$FO\text{mod} \quad \begin{array}{l} t ::= c,\ c \in \mathbb{N} \mid min \mid max \mid x,\ x \in Var \\ \kappa ::= \#x\alpha \sim c, \sim \in \{<,=\} \mid \#x(\alpha) \equiv r(\mod q) \end{array}$$

For example we can define the abbreviated quantifier $Odd\ y\phi$ to stand for $\#y(\phi) \equiv 1(\mod 2)$ and similarly $Even\ y\phi$. A more complicated syntax can be used for allowing a quantifier to do a computation in any finite group (not just a cyclic one). We call this logic FOgrp, the syntax is in [13].

A further restriction FOlen only allows comparison of positions:

$$FO\text{len} \quad \begin{array}{l} t ::= c,\ c \in \mathbb{N} \mid min \mid max \mid x,\ x \in Var \\ \kappa ::= t_1 \sim c, \sim \in \{<,=\} \mid t \equiv r(\mod q) \end{array}$$

Since satisfiability of FO itself is nonelementary over words, it is of interest to study the counting quantifiers in a weaker framework, such as two-variable logics [3, 4, 7, 15].

What is known about the satisfiability problems for these logics over word models? The table gives a status report. The results which are not referenced are available in the Ph.D. thesis [13]. The decidability proofs use a translation to temporal logic [12] and then a small model construction [6]. The undecidability proof is a routine coding.

Two-variable logic	$\lvert\Sigma\rvert = 1$	$\lvert\Sigma\rvert \geq 2$
$FO^2[<, Suc]$		Nexptime[2]
FO^2len$[<, Suc]$		decidable (Nexptime)
FO^2mod$[<, Suc]$		decidable (Expspace)
FO^2grp$[<, Suc]$		decidable (Expspace)
$FO^2[<,+]$		undecidable
FO^2unC$[<, Suc]$	decidable [9, 11]	undecidable [4]

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