RSK bases in invariant theory

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Group action: Examples

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$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} . \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n \end{pmatrix}$$

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There are 2 orbits: $\{\mathbf{0}\}$, $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \neq \mathbf{0}\}$

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$$\begin{pmatrix} \begin{bmatrix} \lambda & 0 & 0 & \cdots \\ 1 & \lambda & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \cdots & 0 & 1 & \lambda \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} \mu & 0 & 0 & \cdots \\ 1 & \mu & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \cdots & 0 & 1 & \mu \end{bmatrix} & 0 & \cdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & . \end{pmatrix}$$

Group actions on varieties

Variety in $\mathbb{C}^n$:	Solutions to a set of polynomials in n-variables in $\mathbb{C}^n$.
Zariski topology on $\mathbb{C}^n$:	Varieties are the closed sets! $\mathbb{C}^n$ **itself is a variety.**
Morphisms:	Maps between varieties given by polynomials. $f : \mathbb{C}^n \to \mathbb{C}$ f is a polynomial in n-variables.

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Earlier example:
$M_n(\mathbb{C}) = \mathbb{C}^{n^2}$ **with Zariski topology.**
$\mathrm{GL}_n(\mathbb{C}) \circlearrowleft M_n(\mathbb{C})$
Each $A \in \mathrm{GL}_n$ gives a morphism. $X \mapsto AXA^{-1}$

Algebraic setting

Corresponding to a variety there is a ring called its *co-ordinate ring*.

The co-ordinate ring of a variety: Ring of morphisms from that variety to $\mathbb{C}$ *(also called regular maps).*

$$\mathbb{C}^n \leftrightarrow \mathbb{C}[x_1, \ldots, x_n].$$

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Variety	$\leftrightarrow$	Ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Co-ordinate ring of a variety in $\mathbb{C}^n$ is $\mathbb{C}[x_1, \ldots, x_n]/I$.
Morphism between varieties	$\leftrightarrow$	Homomorphism of co-ordinate rings.

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A regular map constant on orbits has to be constant on the closure of the orbit!!

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Upshot: It is NOT reasonable to expect a variety structure on $M_n(\mathbb{C})/GL_n$ but rather on orbit closures.

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Classical result

The algebra generated by these elements is the co-ordinate ring of the space of orbit closures.

Multilinear to polynomials

Another procedure: (This generalizes to the case of $GL_n \circlearrowleft M_n(\mathbb{C})^d$)

H.Weyl for characteristic 0, Other characteristics and other classical groups: DeConcini, Procesi, Donkin, Zubkov

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The invariant polynomials are obtained by "restitution" of the invariant multilinear functions.

$$f : M_n \times \cdots \times M_n \to \mathbb{C}$$

$$f(g.(x_1, \ldots, x_d)) = f(x_1, \ldots, x_d) \text{ for } g \in \mathrm{GL}_n$$

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Multi-linear function to invariant polynomial:
Example: Trace $(X_1 X_2) \mapsto$ Trace X^2

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So try to understand the space of multi-linear invariants.

Invariant ring...

Multi-linear invariants on M_n^d: The symmetric group $\mathfrak{S}_d$ gives a set of generators.

$$\mathbb{C}\mathfrak{S}_d \xrightarrow{\Theta} (End(V)^{\otimes d})^* \qquad V := \mathbb{C}^n$$

$[(i_1, i_2, \ldots)(i_k, i_{k+1}, \ldots) \ldots (i_p, i_{p+1}, \ldots) \xrightarrow{\Theta} [A_1, \ldots, A_d) \mapsto$

$\text{Trace}(A_{i_1} A_{i_2} \ldots) \ldots \text{Trace}(A_{i_p} A_{i_{p+1}} \ldots)]$

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Our aim is to get a basis!

Final remarks:

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RSK Basis

The permutations having no decreasing subsequence of length bigger than n gives a basis for the ring of multi-linear invariants of $M_n^d(\mathbb{C})$.

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The proof involves the representation theory of the symmetric group and its Hecke algebra.

It can be posed as a more general problem in the representation theory of symmetric groups, involving tabloids. And the answer to it involves RSK algorithm, hence the name.