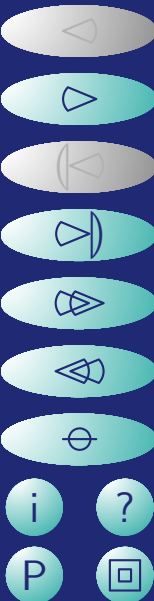


Archana S. Morye

2012-01-08



Aim of the Talk

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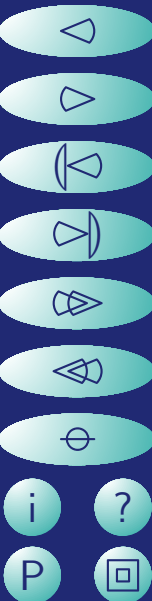


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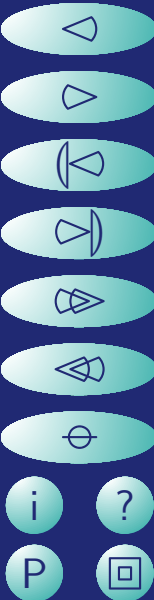
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Vector Bundles

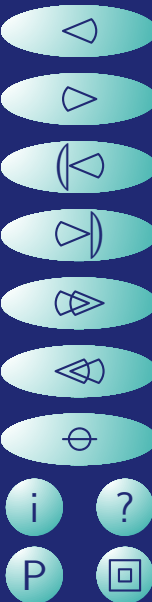
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Example. Consider the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. For every point p in \mathbf{S}^2 , the plane in \mathbf{R}^3 consisting of all vectors which are orthogonal to p is the tangent space $T_p\mathbf{S}^2$ of \mathbf{S}^2 at a point p . Then the tangent bundle $T\mathbf{S}^2 = \coprod_{p \in \mathbf{S}^2} T_p\mathbf{S}^2$ is a vector bundle of rank 2.

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- **Example.** Let X be a \mathcal{C}^∞ manifold, and let $\mathcal{C}_X^\infty(\mathbf{C})$ denote the sheaf of \mathcal{C}^∞ complex valued functions on X , that is, for an open subset U of X

$$\mathcal{C}_X^\infty(\mathbf{C})(U) = \{f : U \rightarrow \mathbf{C} \mid f \text{ is } \mathcal{C}^\infty\}.$$

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- **Definition.** Let (X, \mathcal{O}_X) be a ringed space. We say that an \mathcal{O}_X -module \mathcal{F} is *locally free* if for every $x \in X$, there exist an open neighborhood U of x , and a set I such that $\mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$ as an $\mathcal{O}_X|_U$ -module.

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- Let X be a connected manifold. Then, the category of K -vector bundles on X ($K = \mathbf{R}$ or \mathbf{C}), and the category of locally free sheaf of finite rank are equivalent categories.

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- **Definition.** Let E be a \mathcal{C}^∞ complex vector bundle of rank r over X . A \mathcal{C}^∞ *connection* ∇ in E is a \mathbf{C} -linear sheaf morphism,

$$\nabla : A^0(E) \longrightarrow A^1(E)$$

which satisfies the Leibnitz identity, $\nabla(fs) = f\nabla(s) + df \cdot s$, for $f \in A^0$, $s \in A^0(E)$, where $A^p(E)$ denotes the sheaf of \mathcal{C}^∞ p -forms with values in E .

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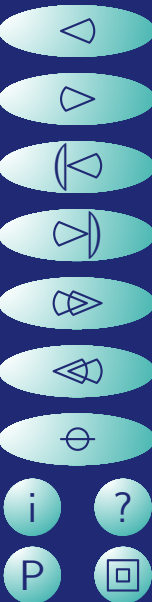
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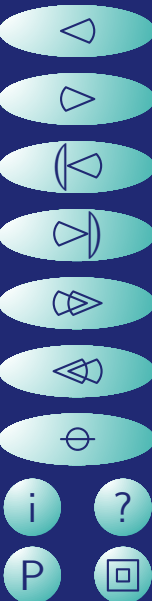
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- If the connection is flat, then a parallel transport is invariant under smooth homotopies.
- A vector bundle admits a flat connection if and only if it is defined by a representation of the fundamental group $\rho : \pi_1 \rightarrow \mathrm{GL}(r, \mathbf{C})$.



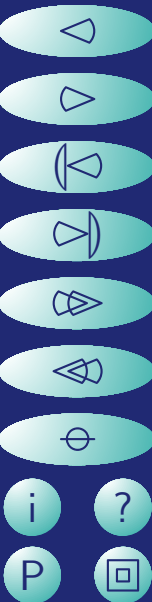
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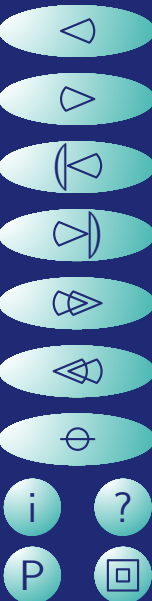
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- But the converse is not true in general.



Stable Vector Bundles

Let X be a Kähler manifold, and let Φ be its Kähler form. Then for any vector bundle E over X ,

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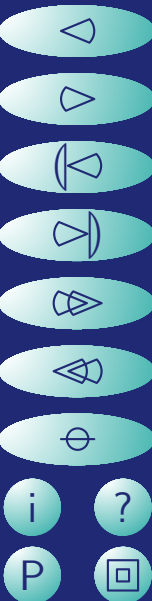
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One can extend the definition of a degree for torsion free coherent sheaves.

Definition. A holomorphic vector bundle E over a compact Kähler manifold is said to be *stable* (respectively *semistable*) if for every proper holomorphic coherent subsheaf \mathcal{F} with $0 < \text{rank}(\mathcal{F}) < \text{rank}(E)$, we have

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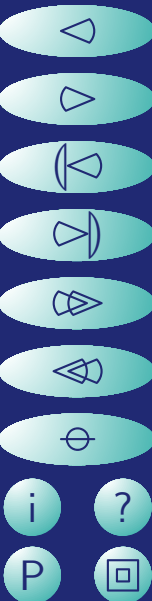
$$\text{where } \mu(\mathcal{F}) = \frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})}.$$



Stable Vector Bundles (Continued)

Stable vector bundles are important in physics, differential geometry. Stable vector bundles over Riemann surfaces are closely related to Yang-Mills theory. Narasimhan-Seshadri Theorem give this correspondence.

Theorem. (Narasimhan-Seshadri) A stable holomorphic vector bundle over a Riemann surface admits a Einstein-Hermitian metric and conversely.



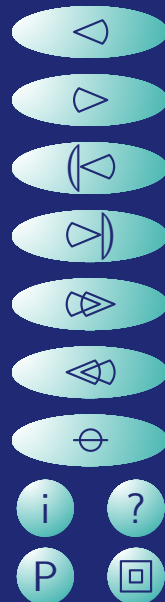
Vector Bundles over Real Abelian Variety

- **Definition.** A *real abelian variety* is a real holomorphic manifold (X, σ) , where the underlying complex manifold is an abelian variety, and the antiholomorphic involution σ is compatible with the group operation, that is, $\sigma(x + y) = \sigma(x) + \sigma(y)$ for all $x, y \in X$.



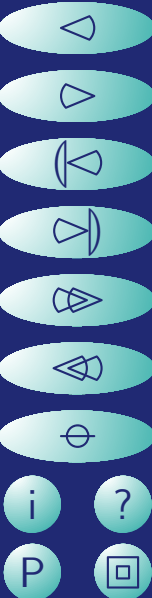
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- A vector bundle over an abelian variety is called *homogeneous* if it is invariant under all translations.



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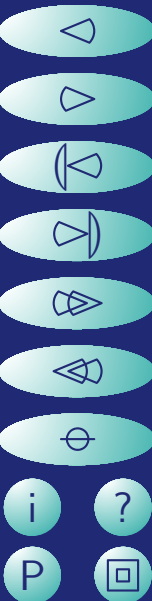
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- **Definition.** If for all real point $x \in X$ (that is, $\sigma(x) = x$), $(\tau_x^*(E), \alpha^{\tau_x^*(E)})$ is isomorphic to (E, α^E) in the category of \mathcal{O}_X -**mod**^{real}, then (E, α^E) is said to be *real homogeneous*, where $\tau_x : X \rightarrow X, y \mapsto y + x$ is the *translation of X by x*

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Vector Bundles over Real Abelian Variety (Continued)

- A *real holomorphic connection* in a real holomorphic vector bundle is a holomorphic connection, which is compatible with the real structure on E .
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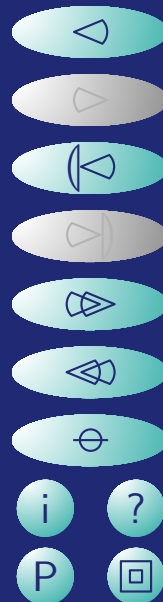
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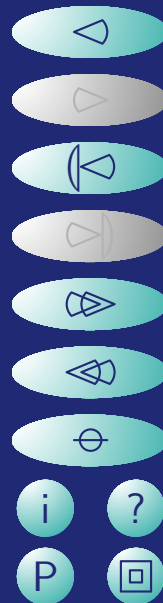
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$$E^\bullet : \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that E_i is a real sub-bundle of (E, α^E) , $c_j(E_i) = 0$, for $j = 1, 2$ and $i = 1, \dots, n$, and E_i/E_{i-1} is real polystable.

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5. (E, α^E) admits a real flat holomorphic connection.