Vector Bundles over Real Abelian Varieties

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Aim of the Talk

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Example. Consider the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. For every point p in $\mathbf{S}^2$, the plane in $\mathbf{R}^3$ consisting of all vectors which are orthogonal to p is the tangent space $T_p\mathbf{S}^2$ of $\mathbf{S}^2$ at a point p. Then the tangent bundle $T\mathbf{S}^2 = \coprod_{p \in S^2} T_p\mathbf{S}^2$ is a vector bundle of rank 2.

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$$\mathcal{C}_X^\infty(\mathbf{C})(U) = \{f : U \to \mathbf{C} \mid f \text{ is } \mathcal{C}^\infty\}.$$

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- **Definition.** Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if for every $x \in X$, there exist an open neighborhood U of x, and a set I such that $\mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$ as an $\mathcal{O}_X|_U$-module.

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- Let X be a connected manifold. Then, the category of K-vector bundles on X ($K = \mathbf{R}$ or $\mathbf{C}$), and the category of locally free sheaf of finite rank are equivalent categories.

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- **Definition.** Let E be a $\mathcal{C}^\infty$ complex vector bundle of rank r over X. A $\mathcal{C}^\infty$ *connection* ∇ in E is a $\mathbf{C}$-linear sheaf morphism,

$$\nabla : A^0(E) \longrightarrow A^1(E)$$

which satisfies the Leibnitz identity, $\nabla(fs) = f\nabla(s) + df \cdot s$, for $f \in A^0$, $s \in A^0(E)$, where $A^p(E)$ denotes the sheaf of $\mathcal{C}^\infty$ p-forms with values in E.

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- We say a $\mathcal{C}^\infty$ (respectively holomorphic) connection is *flat* if $\nabla^2 = 0$ (respectively $D^2 = 0$).

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- **Definition.** Let E be a $\mathcal{C}^\infty$ complex vector bundle over X equipped with a $\mathcal{C}^\infty$-connection ∇, and a smooth curve $\alpha : [0,1] \to X$, with $\alpha(0) = a$, $\alpha(1) = b$. Then, there is an induced $\mathbf{C}$-linear map $P_\alpha : E_a \to E_b$ called a *parallel transport* operator.

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- If the connection is flat, then a parallel transport is invariant under smooth homotopies.
- A vector bundle admits a flat connection if and only if it is defined by a representation of the fundamental group $\rho : \pi_1 \to \mathrm{GL}(r, \mathbf{C})$.

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- Let X be a connected complex manifold of Kähler type, and E be a holomorphic vector bundle over X. If E admits a holomorphic connection, then all $c_j(E) \in \mathrm{H}^{2j}(X, \mathbf{C}) = 0$, for $j > 0$.

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- But the converse is not true in general.

Stable Vector Bundles

Let X be a Kähler manifold, and let Φ be its Kähler form. Then for any vector bundle E over X,

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Definition. A holomorphic vector bundle E over a compact Kähler manifold is said to be *stable* (respectively *semistable*) if for every proper holomorphic coherent subsheaf $\mathcal{F}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(E)$, we have

$$\mu(\mathcal{F}) < \mu(E) \quad (\text{respectively } \mu(\mathcal{F}) \leq \mu(E)),$$

where $\mu(\mathcal{F}) = \frac{\text{degree}(\mathcal{F})}{\text{rank}(\mathcal{F})}$.

Stable Vector Bundles (Continued)

Stable vector bundles are important in physics, differential geometry. Stable vector bundles over Riemann surfaces are closely related to Yang-Mills theory. Narasimhan-Seshadri Theorem give this correspondence.

Theorem. (Narasimhan-Seshadri) A stable holomorphic vector bundle over a Riemann surface admits a Einstein-Hermitian metric and conversely.

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- **Definition.** A *real abelian variety* is a real holomorphic manifold (X, σ), where the underlying complex manifold is an abelian variety, and the antiholomorphic involution σ is compatible with the group operation, that is, $\sigma(x+y) = \sigma(x) + \sigma(y)$ for all $x, y \in X$.

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- **Definition.** If for all real point $x \in X$ (that is, $\sigma(x) = x$), $(\tau_x^*(E), \alpha^{\tau_x^*(E)})$ is isomorphic to (E, α^E) in the category of $\mathcal{O}_X$-$\mathbf{mod}^{\mathbf{real}}$, then (E, α^E) is said to be *real homogeneous*, where $\tau_x : X \to X$, $y \mapsto y + x$ is the *translation of* X *by* x

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2. (E, α^E) *is real homogeneous.*
3. (E, α^E) *is real semistable with* $c_1(E) = c_2(E) = 0$.
4. (E, α^E) *admits a filtration*

$$E^\bullet : \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that E_i *is a real sub-bundle of* (E, α^E), $c_j(E_i) = 0$, *for* $j = 1, 2$ *and* $i = 1, \ldots, n$, *and* E_i / E_{i-1} *is real polystable.*

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5. *(E, α^E) admits a real flat holomorphic connection.*