CHERN-SIMONS CLASSES OF FLAT CONNECTIONS ON SUPERMANIFOLDS

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Abstract. In this note we define the Chern-Simons classes of a flat superconnection $D+L$ on a complex supervector bundle E such that D preserves the grading, and L is an odd endomorphism of E on a supermanifold. As an application we obtain a definition of Chern-Simons classes of a (not necessarily flat) morphism between flat vector bundles on a smooth manifold. We extend Reznikov's theorem on triviality of these classes when the manifold is a compact Kähler manifold or a smooth complex quasi–projective variety, in degrees > 1.

1. Introduction

Suppose $(X, \mathcal{C}^\infty_X)$ is a $\mathcal{C}^\infty$-differentiable manifold endowed with the structure sheaf $\mathcal{C}^\infty_X$ of smooth functions. Let E be a complex $\mathcal{C}^\infty$ vector bundle on X of rank r and equipped with a connection ∇. The Chern-Weil theory defines the Chern classes

$$c_i(E,\nabla) \in H^{2i}_{dR}(X,\mathbb{C}), \text{ for } i = 0, 1, ..., r$$

in the de Rham cohomology of X. These classes are expressed in terms of the GL_r-invariant polynomials evaluated on the curvature form ∇^2.

Suppose E has a flat connection, i.e., $\nabla^2 = 0$. Then the de Rham Chern classes are zero. It is significant to define Chern-Simons classes for a flat connection. These are classes in the $\mathbb{C}/\mathbb{Z}$-cohomology and were defined by Chern-Cheeger-Simons in [6], [7].

Quillen has pointed out in [19],[20], a homomorphism $u : E_0 \to E_1$ between vector bundles on a smooth manifold M and inducing an isomorphism over a subset $A \subset M$ corresponds to an element in the relative K-group $K(M, A)$. A Chern character in the de Rham cohomology of M associated to the homomorphism u is computed in [19] whose class is shown to be equal to the difference $\mathrm{ch}(E_0) - \mathrm{ch}(E_1)$ of the Chern characters. This describes the Chern character of the homomorphism u. In fact, we think that it would be good to look at a quiver, i.e., a sequence of homomorphisms between vector bundles

$$E_0 \to E_1 \to ... \to E_r$$

over a smooth manifold and define the Chern character of the sequence in the de Rham cohomology. This will involve a study of $\mathbb{Z}_{r+1}$-graded objects, which we will look in the future. Quillens proof involves regarding $E = E_0 \oplus E_1$ as a supervector bundle on M and

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D be any connection preserving the grading and associating an odd endomorphism of E, with respect to u and a choice of a metric.

In this paper, we want to look at a morphism u between flat vector bundles and extend Quillen's construction and define Chern-Simons classes for the morphism u. Hence it is relevant to define Chern-Simons classes for flat connections in the setting of supermanifolds, in a more general set-up.

For the definition of *supermanifolds*, see [8] (as well as [1], [15]). The Chern classes of supervector bundles are defined in [4], on a supermanifold in the integral cohomology. We note that the usual Chern-Weil theory on smooth manifolds expresses de Rham Chern classes in terms of GL_n-invariant polynomials on the curvature form of a connection on a smooth vector bundle. In the supersetting, a study of the $GL(r,s)$–invariant polynomials has been carried out by Sergeev [23], following works by Berezin [3],[5] and Kac [12], see also [24] by Shander. The differential forms defined by Quillen which are obtained from the Chern character $\mathrm{str}e^{D+L}$ can be expressed as rational functions of the $GL(r,s)$–invariant polynomials, by the above results in [3], [5], [12], [24], [23]. In this paper we use the existence of such polynomials to define the Chern-Simons classes.

Let $(M,\mathcal{O}_M)$ denote a complex supermanifold and $(M_B,\mathcal{C}^\infty_M)$ denote the underlying $\mathcal{C}^\infty$-manifold.

With notations as in [8] or §2, we show

Theorem 1.1. *Suppose $\{\nabla_t\}_t$ is a family of superconnections on a complex supervector bundle E, such that ∇_0 preserves the grading. Suppose ∇_{t_0} is flat, for some t_0. Then there is a uniquely determined Chern-Simons class*

$$\widehat{c_n}(E,D_{t_0})\in H^{2n-1}(M_B,\mathbb{R}/\mathbb{Z}),$$

for $n\geq 1$.

In particular this applies to the following situation:

Corollary 1.2. *Suppose $(M,\mathcal{O}_M)$ is a complex supermanifold. Let $\mathcal{E}^{r|s}$ be a complex supervector bundle on $(M,\mathcal{O}_M)$ equipped with a superconnection $\nabla = D+L$ such that D preserves the grading and L is an odd endomorphism of $\mathcal{E}^{r|s}$. Assume that ∇ is a flat superconnection. Then there exists uniquely determined Chern-Simons classes*

$$\widehat{c_n}(\mathcal{E}^{r|s},\nabla)\ \in\ H^{2n-1}(M_B,\mathbb{C}/\mathbb{Z})$$

for $n>0$. Furthermore, if M_B is a compact Kähler manifold or a smooth complex quasi–projective variety and D itself is a flat smooth connection, then these classes are torsion, in degrees >1.

This can be thought of as an extension of Reznikov's fundamental theorem [22] on rationality of Chern-Simons classes on compact Kähler manifold, in the setting of supermanifolds. We also define Chern-Simons classes of a (not necessarily flat) homomorphism

$u : E_0 \to E_1$ between flat complex vector bundles, extending Quillen's construction of the de Rham Chern character. Then we prove a relative Reznikov theorem (see Theorem 3.12) for the classes of the morphism u. More generally, we extend the question of Cheeger-Simons on the rationality of these classes (see Question 3.10) for flat superconnections of the type $D + L$.

2. Preliminaries

We briefly recall the definitions and terminologies from [15] and from the notes by Deligne and Morgan [8].

Let $\mathcal{C}^\infty$ be the sheaf of C^∞-functions on $\mathbb{R}^p$. The space $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$, endowed with the sheaf $\mathcal{C}^\infty[\theta_1, ..., \theta_q]$ of supercommutative super $\mathbb{R}$-algebras, freely generated over $\mathcal{C}^\infty$ by the anticommuting $\theta_1, ..., \theta_q$. The coordinates t^i of $\mathbb{R}^p$ and the θ_j and all generators of $\mathcal{C}^\infty$ obtained from them by any automorphism are said to be the coordinates of $\mathbb{R}^{p|q}$. A supermanifold M of dimension $p|q$ is a topological space M_B (or also called as the body manifold with the structure sheaf $\mathcal{C}^\infty_M$) endowed with a sheaf of super $\mathbb{R}$-algebras which is locally isomorphic to $\mathbb{R}^{p|q}$. The structure sheaf of M is denoted by $\mathcal{O}_M$. We denote $p|q$, the real dimension of the supermanifold M.

On $M = \mathbb{R}^{p|q}$, the even derivations $\partial/\partial t^i$ and the odd derivations $\partial/\partial\theta^j$ are defined.

Proposition 2.1. [15, 2.2.3] *The $\mathcal{O}_M$-module of $\mathbb{R}$-linear derivations of $\mathcal{O}_M$ is free of dimension $p|q$, with basis: the $\partial/\partial t^i$ and the $\partial/\partial\theta^j$.*

Complex supermanifolds are topological spaces endowed with a sheaf of super $\mathbb{C}$-algebras, locally isomorphic to some $(\mathbb{C}^p, \mathcal{O}[\theta^1, ..., \theta^q])$. Here $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^p$. As before we denote $p|q$, the complex dimension of the complex supermanifold M.

Suppose R be a commutative superalgebra and the standard free module $A^{r|s}$ is the module freely generated by even elements $e_1, ..., e_r$ and odd elements $f_1, ..., f_s$. An automorphism of $A^{r|s}$ is represented by an invertible matrix

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \tag{1}$$

such that the $(r\times r)$–matrix X_1 and the $(s\times s)$–matrix X_4 have even entries and the $(s\times r)$–matrix X_3 and the $(r \times s)$–matrix X_2 have odd entries. The group of all automorphisms of $A^{r|s}$ is denoted by $GL(r, s)$.

The *supertrace* of the matrix X is the difference

$$\mathrm{str(X) := tr(X_1) - tr(X_2)}$$

of the usual trace of the matrices X_1 and X_4.

Suppose M is a supermanifold and locally it looks like $\mathbb{R}^{p|q}$ as above. A *complex supervector bundle* V on M is a fiber bundle V over M with typical fiber $\mathbb{C}^{r|s}$ and structural group $GL(r,s)$. Alternately, it can be considered as a sheaf of $\mathcal{O}_M$-supermodules $\mathcal{V}$, locally free of rank $r|s$.

The tangent bundle $\mathcal{T}_M$ is the $\mathcal{O}_M$-module of derivations of $\mathcal{O}_M$ and is a supervector bundle of rank $p|q$. The cotangent bundle Ω^1_M is the dual of $\mathcal{T}_M$. There is a differential $d : \mathcal{O}_M \longrightarrow \Omega^1_M$, giving rise to the super de Rham complex $\Omega^\bullet_M$ on M.

Lemma 2.2. (Poincaré lemma)[8, p.73] *The complex $\Omega^\bullet_M$ is a resolution of the constant sheaf on the body manifold M_B.*

In particular, the cohomology of M_B can be computed by the super de Rham complex:

$$H^*(M_B,\mathbb{R}) = H^*(\Gamma(M,\Omega^\bullet_M)).$$

We briefly review the group of differential characters and Chern-Simons classes on a smooth manifold X.

2.1. **Analytic differential characters on** X [6]**.** Let $S_k(X)$ denote the group of k-dimensional smooth singular chains on X, with integer coefficients. Let $Z_k(X)$ denote the subgroup of cycles. Let us denote

$$S^\bullet(X,\mathbb{Z}) := \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X),\mathbb{Z})$$

the complex of $\mathbb{Z}$ -valued smooth singular cochains, whose boundary operator is denoted by δ. The group of smooth differential k-forms on X with complex coefficients is denoted by $A^k(X)$ and the subgroup of closed forms by $A^k_{cl}(X)$. Then $A^\bullet(X)$ is canonically embedded in $S^\bullet(X)$, by integrating forms against the smooth singular chains. In fact, we have an embedding

$$i_{\mathbb{Z}} : A^\bullet(X) \hookrightarrow S^\bullet(X,\mathbb{C}/\mathbb{Z}).$$

The group of differential characters of degree k is defined as

$$\widehat{H^k}_{\mathbb{C}}(X) := \{(f,\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(X),\mathbb{C}/\mathbb{Z}) \oplus A^k(X) : \delta(f) = i_{\mathbb{Z}}(\alpha) \text{ and } d\alpha = 0\}.$$

There is a canonical and functorial exact sequence:

$$0 \longrightarrow H^{k-1}(X,\mathbb{C}/\mathbb{Z}) \longrightarrow \widehat{H^k}_{\mathbb{C}}(X) \longrightarrow A^k_{\mathbb{Z}}(X) \longrightarrow 0. \tag{2}$$

Here $A^k_{\mathbb{Z}}(X) := \ker(A^k_{cl}(X) \longrightarrow H^k(X,\mathbb{C}/\mathbb{Z}))$.

Similarly, one can define the group of differential characters $\widehat{H^k}_{\mathbb{R}}(X)$ which have $\mathbb{R}/\mathbb{Z}$-coefficients.

2.2. **Cheeger-Chern-Simons classes.** Suppose (E,θ) is a vector bundle with a connection on X. Then the characteristic forms

$$c_k(E,\theta) \in A^{2k}_{cl}(X,\mathbb{Z})$$

for $0 \leq k \leq r$=rank (E), are defined using the universal Weil homomorphism [7].

The characteristic classes

$$\widehat{c_k}(E,\theta) \in \widehat{H^{2k}}_{\mathbb{C}}(X)$$

are defined in [6] using a factorization of the universal Weil homomorphism. These classes are functorial lifting of the forms $c_k(E,\theta)$.

Similarly, there are classes

$$\widehat{c_k}(E,\theta) \in \widehat{H^{2k}}_{\mathbb{R}}(X).$$

Remark 2.3. *If the forms $c_k(E,\theta)$ are zero, then the classes $\widehat{c_k}(E,\theta)$ lie in the cohomology $H^{2k-1}(X,\mathbb{C}/\mathbb{Z})$. If (E,θ) is a flat bundle, then $c_k(E,\theta)=0$ and the classes $\widehat{c_k}(E,\theta)$ are called as the Chern-Simons classes of (E,θ). Notice that the class depends on the choice of θ.*

Beilinson's theory of universal Chern-Simons classes yield classes for a flat connection (E,θ),

$$\widehat{c_k}(E,\theta) \in H^{2k-1}(X,\mathbb{C}/\mathbb{Z})$$

which are functorial and additive over exact sequences (see [9] and [10] for another construction).

3. Chern-Simons classes of flat superconnections on supermanifolds

Let $(M,\mathcal{O}_M)$ be a complex supermanifold of dimension $p|q$. Consider the sheaf of differentials Ω^1_M on M and let $\mathcal{E}^{r|s}$ be a complex supervector bundle on M of rank $r|s$.

Lemma 3.1. *Given a complex supervector bundle $\mathcal{E}^{r|s}$ of rank $r|s$ on M, there exists a direct sum decomposition*

$$E = E_0 \oplus E_1$$

for some complex smooth vector bundles E_0 and E_1 of rank r and rank s respectively, on the underlying $\mathcal{C}^\infty$-manifold M_B.

Proof. A rank $r|s$ complex supervector bundle $\mathcal{E}^{r|s}$ determines two complex smooth bundles E_0 and E_1 on the underlying smooth manifold M_B of M as follows. One considers the body map

$$\mathcal{O}_M \longrightarrow \mathcal{C}^\infty_M \otimes \mathbb{C}$$

which is obtained by forgetting the local anticommuting variables θ_j. Let $\overline{\mathcal{E}^{r|s}}$ denote the sheaf of (super)sections of $\mathcal{E}^{r|s}$. Then $\overline{E_{r+s}} := \overline{\mathcal{E}^{r|s}} \otimes_{\mathcal{O}_M} (\mathcal{C}^\infty_M \otimes \mathbb{C})$ is the sheaf of sections of a rank $r+s$ smooth complex vector bundle E_{r+s} on the body manifold M_B. Locally, the sheaf $\mathcal{E}^{r|s}$ is generated by r even elements and s odd elements as a $\mathcal{O}_M = \mathcal{C}^\infty_M[\theta_1,...,\theta_q]$–module. Hence tensoring with $\mathcal{C}^\infty_M$ locally gives a rank $r+s$ free $\mathcal{C}^\infty_M$–module given by the same generators. This implies that the complex vector bundle E_{r+r} is of rank $r+s$. Now, we notice that the structural group of $\mathcal{E}^{r|s}$ is $GL(r,s)$ and the structural group of the vector bundle E_{r+s} factors via the projection

$$GL(r,s) \to GL(r+s).$$

Using the description of the elements in $GL(r,s)$ in (1), it follows that the image under this projection consists of block diagonal matrices of sizes $r \times r$ and $s \times s$.

This implies that the matrix of the transition functions of E_{r+s} is a block diagonal matrix of rank r and rank s which correspond to smooth complex vector bundles E_0 and E_1 such that $r = \text{rank } E_0$ and $s = \text{rank } E_1$ on M_B. □

In view of the above lemma, we may regard a supervector bundle $\mathcal{E}^{r|s}$ on M as a supervector bundle $E = E_0 \oplus E_1$, on the underlying $\mathcal{C}^\infty$-manifold M_B where E_0 and E_1 are $\mathcal{C}^\infty$-vector bundles on M_B.

3.1. **Superconnections.** Let $\mathcal{E}^{r|s} = E = E_0 \oplus E_1$ be a complex supervector bundle on a manifold M_B. Let $\Omega(M_B) = \oplus \Omega^p(M_B)$ be the algebra of smooth differential forms with complex coefficients. Let

$$\Omega(M_B, E) := \Omega(M_B) \otimes_{\Omega^0(M_B)} \Omega^0(M_B, E).$$

where $\Omega^0(M_B, E)$ is the space of (super)sections of E.

Then $\Omega(M_B, E)$ has a grading with respect to $\mathbb{Z} \times \mathbb{Z}_2$

A *superconnection* D on $\mathcal{E}^{r|s}$ is an operator on $\Omega(M_B, E)$ of odd degree satisfying the derivation property

$$D(\omega\alpha) = (d\omega)\alpha + (-1)^{deg\,\omega}\nabla\alpha.$$

For example, a connection on E preserving the grading when extended to an operator on $\Omega(M_B, E)$ in the usual way determines a superconnection.

In local coordinates, when E is trivial it looks like $M_B \times V$, V is a $\mathbb{Z}_2$-graded complex vector space, and a superconnection D is of the form $d + \theta$, where θ is an odd element of $\Omega(M_B) \otimes \text{End}(V)$.

The *curvature* of a superconnection D is the even degree operator $D^2 := D \circ D$ on $\Omega(M_B, E)$.

A superconnection is said to be *flat* if $D^2 = 0$. We call the pair $(\mathcal{E}^{r|s}, D)$ as a flat complex supervector bundle.

We want to define Chern-Simons classes of (E, D) when D is a flat superconnection, for special types of superconnection.

For this purpose, we look at the situation, considered by Quillen [19] when the superconnection is locally of the form $d + \theta$ where

$$\theta = A + L \in \Omega^1(M_B) \otimes (\text{End } V)^0 \oplus \Omega^0(M_B) \otimes (\text{End } V)^1.$$

It is an interesting question to define Chern-Simons classes for arbitrary flat superconnections $d+\theta$, where θ is an arbitrary odd element of $\Omega(M_B) \otimes \text{End}(V)$, which we do not know how to treat.

3.2. **Quillen's construction.** Suppose M is a supermanifold and E is a complex supervector bundle on M. Regard $E = E_0 \oplus E_1$ as a complex supervector bundle on M in view of Lemma 3.1, where E_0 and E_1 are smooth vector bundles on the body manifold M_B. Under this identification we omit the suffix B from M_B and without confusion we write $M = M_B$ in the following discussion.

Suppose E is equipped with a superconnection D. From the curvature D^2, one can construct differential forms

$$\mathrm{str}(D^2)^n = \mathrm{str} D^{2n}$$

in $\Omega(M)^{even}$. These are even forms since the supertrace preserves the grading.

We have,

Theorem 3.2. *The form* $\mathrm{strD^{2n}}$ *is closed, and its de Rham cohomology class is independent of the choice of superconnection D.*

Proof. See [19, Theorem, p.91]. □

Quillen described the (super) Chern character of E in terms of the usual Chern characters of E_0 and E_1 in the following situation.

Regard $E = E_0 \oplus E_1$ as a complex supervector bundle and $D = D^0 + D^1$ be a connection on E preserving the grading. Let L be an odd degree endomorphism of E and write $D_t := D + t.L$ where t is a parameter.

Proposition 3.3. (Quillen)[19] *Replacing L by $t.L$, where t is a parameter, one obtains a family of forms*

$$\mathrm{str}\,\mathrm{e}^{(\mathrm{D+tL})^2} = \mathrm{str}\,\mathrm{e}^{\mathrm{r^2L^2+t[D,L]+D^2}} \tag{3}$$

all of which represent the Chern character $ch(E_0) - ch(E_1)$ in the de Rham cohomology of M. Here str *denotes the supertrace.*

□

The referee has pointed out that the above computations on a supermanifold produces pseudodifferential forms. For our purpose, it suffices to note that the trace form in (3) defines a closed differential form whose de Rham class is independent of the superconnection [19, Theorem p.91].

In this situation we want to define uniquely determined Chern-Simons classes of (E, D_t) which is independent of t and if $\nabla_1 = D + L$ is flat.

For this purpose, we look at the *Character diagram* of Simons and Sullivan (see [25]):

$$
\begin{tikzcd}
0 \arrow[drr] & & & & & & 0 \\
{} \arrow[dr] & & H^{k-1}(\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "i_1"] & & H^k(\mathbb{Z}) \arrow[urr] \arrow[dr, "r"] & & {} \\
& H^{k-1}(\mathbb{R}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & \widehat{H^k}_{\mathbb{R}}(M) \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^k(\mathbb{R}) \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & A^{k-1}/A^{k-1}_{\mathbb{Z}} \arrow[rr, "d"] \arrow[ur, "i_2"] & & A^k_{\mathbb{Z}} \arrow[drr] \arrow[ur, "s"] & & {} \\
0 \arrow[urr] & & & & & & 0
\end{tikzcd}
\tag{4}
$$

The diagonal sequences are exact and (α, B, r) is the Bockstein long exact sequence associated to the coefficient sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Also (β, d, s) is another long exact sequence in which β and s are defined via the de Rham theorem. (A similar diagram holds by replacing $\mathbb{R}$ with $\mathbb{C}$).

Lemma 3.4. *Suppose (F, ∇) is a smooth connection on a manifold M. Then there is a uniquely determined differential character*

$$\widehat{c_k}(F, \nabla) \in \widehat{H^k}_{\mathbb{R}}(M)$$

which lifts the k-th Chern form defined in $A^k_{\mathbb{Z}}$. Furthermore, if ∇ is flat then $c_k(F) \in H^{2k}(M, \mathbb{Z})$ vanishes in $H^{2k}(M, \mathbb{R})$. There is a unique lifting $\widehat{c_k}(F, \nabla) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ of the integral class $c_k(F)$, for $k > 0$.

Proof. The vanishing of the Chern form for a flat connection is by the Chern-Weil theory. The rest of the assertion is by the Chern-Simons-Cheeger construction [6] of differential characters. □

Consider the total Chern class

$$c(F) = 1 + c_1(F) + ... + c_f(F)$$

and the total Segre class

$$s(F) = 1 + s_1(F) + ... + s_f(F).$$

Then we have the relations

$$s(F) = \frac{1}{c(F)} \tag{5}$$

and

$$c(F - G) = c(F).s(G) \tag{6}$$

where G is any other vector bundle. These relations also hold if we replace the classes $c_i(F)$ by $\widehat{c_i}(F, \nabla)$ and $s_i(F)$ by $\widehat{s_i}(F, \nabla)$ which are defined by the relation (5). See [6, p.64-65].

Our goal is to define a canonical lifting in $\widehat{H^*_{\mathbb{R}}}(M)$ of the supertrace form (3) associated to a superconnection (E, ∇).

To motivate the definition, we firstly look at the superconnection of the type $D + L$ where D preserves the grading and L is an odd endomorphism of the complex supervector bundle E. We consider the family of superconnection $D_t = D + t.L$ as above. We will see how the class is represented in the de Rham cohomology. Consider the product manifold $\mathbb{R} \times M$ and the pullback $pr_2^* E$ of the bundle E. This bundle is equipped with a superconnection

$$\bar{D} := dt\partial_t + D'$$

whose restriction to $\{t\} \times M$ is D_t. In terms of local trivialization of $E = M \times V$ we can describe $\bar{D}$, D' as follows. Write $D_t = d_M + \theta_t$, where θ_t is a family of one-forms on M with values in $\mathrm{End}V$ and let θ be the form on $\mathbb{R} \times M$ not involving dt and having the restriction θ_t on $\{t\} \times M$. Then $D' = d_M + \theta$ and

$$\bar{D} = dt\partial_t + D' = (dt\partial_t + d_M) + \theta = d_{\mathbb{R}\times M} + \theta.$$

See also [19, p.91].

By the homotopy property of de Rham cohomology, it follows that the class of $\mathrm{str} D_t^{2n}$ in $H^{2n}(M, \mathbb{R})$ is independent of t.

Proposition 3.5. *Suppose the superconnection $D = D^0 \oplus D^1$ on the supervector bundle $E = E_0 \oplus E_1$ preserves the grading and the individual connections D^0 and D^1 are smooth flat connections on E_0 and E_1 respectively. Then $D_t = D + t.L$ is a superconnection on E where L is an odd endomorphism of E. Then there is a uniquely determined class $\widehat{c_n}(E, D_t) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$, independent of t. Moreover this class is equal to*

$$\widehat{c_n}(E, D + L) = \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1)$$

Proof. We notice that the trace forms are integral valued. This implies that the Chern class associated to these forms lies in the integral cohomology $H^{2n}(M, \mathbb{Z})$ which is independent of t in $H^{2n}(M, \mathbb{R})$, by Quillen's Theorem 3.2. This determines a class in $H^{2n}(M, \mathbb{Z})$ independent of t. But this class vanishes in $H^{2n}(M, \mathbb{R})$ since D has components D^0 and D^1 which are flat connections, hence $D^2 = 0$. Using the Bockstein operator in (4), we conclude that there is a class

$$\widehat{c_n}(E, D_t) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$$

which is independent of t and we denote this class by $\widehat{c_n}(E, D+L)$. We get a uniquely determined class $\widehat{c_n}(E, D+L) = \widehat{c_n}(E, D) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$, as follows.

To get an expression for this class, we notice that by Quillen's result Proposition 3.3 the (super) Chern character form of E is the difference $\mathrm{ch}(E_0) - \mathrm{ch}(E_1)$ in the integral cohomology. In particular we want to lift the integral Chern classes of $E_0 - E_1$ in the $\mathbb{R}/\mathbb{Z}$-cohomology. The relations in (5) and (6) give the formula

$$\widehat{c_n}(E, D+L) := \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1).$$

The uniqueness of $\widehat{c_n}(E, D+L)$ follows from the uniqueness of the Chern-Simons classes $\widehat{c_p}(E_0, D^0)$ and $\widehat{c_q}(E_1, D^1)$, see Lemma 3.4. This concludes the lemma. $\square$

Remark 3.6. *All the above constructions follow verbatim by replacing $\mathbb{R}/\mathbb{Z}$-coefficients with $\mathbb{C}/\mathbb{Z}$-coefficients. We call the resulting classes $\widehat{c_n}(E, D+L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$ as the Chern-Simons classes of the superconnection $D+L$ (or $D+t.L$ for a parameter t).*

To define the Chern-Simons class for any flat superconnection, we consider the (super) Chern character of $D+L$,

$$\mathrm{ch(D+L) = str\ e^{(D+L)^2}}.$$

We look at the degree $2i$-terms of this expression,

$$\mathrm{ch_i(D+L) := (str\ e^{(D+L)^2})_{2i}}$$

and we want to lift $\mathrm{ch_i(D+L)}$ as a differential character. Notice that there is a $GL(r,s)$-invariant polynomial P_i (see [3], [5], [12], [24], [23]) such that the term $\mathrm{ch_i(D+L)}$ is obtained by plugging in $D+L$ in P_i, i.e.,

$$\mathrm{ch_i(D+L) = P_i(D+L, ..., D+L)}. \tag{7}$$

Lemma 3.7. *Suppose ∇_t is a family of superconnections on a complex supervector bundle E such that when $t=0$, ∇_0 is a connection which preserves the grading. Then we can define the n-th Chern class of (E, ∇_t) (equivalently $\widehat{\mathrm{ch_n}}(\nabla_t)$) in the ring of differential characters.*

Proof. Firstly, since ∇_0 preserves the grading on E, it corresponds to smooth connections D_0 and D_1 on the component bundles of $E = E_0 \oplus E_1$. Hence the differential character $\widehat{c_n}(E, \nabla_0)$ is defined by the expression (see Proposition 3.5),

$$\widehat{c_n}(E, \nabla_0) = \sum_{p+q=n} \widehat{c_p}(E_0, D_0).\widehat{s_q}(E_1, D_1).$$

Similarly, we can define the n-degree term of the Chern character in terms of the Chern characters of E_0 and E_1,

$$\widehat{\mathrm{ch}_n}(E, \nabla_0) := \widehat{\mathrm{ch}_n}(E_0, \nabla_0) - \widehat{\mathrm{ch}_n}(E, \nabla_0).$$

To define $\widehat{\mathrm{ch}_n}(E,\nabla_t)$, for $t \neq 0$, we can use the variational formula of differential characters of Cheeger-Simons [6, Proposition], obtained by looking at the polynomial P_n which defines $\widehat{\mathrm{ch}_n}(E,\nabla_1)$ (see (7)):

$$\widehat{\mathrm{ch}_n}(E,\nabla_1) := n.\int_0^1 P_n(\frac{d}{dt}\nabla_t \wedge \nabla_t^{2(n-1)})dt]_{Z_{2n-1}} + \widehat{\mathrm{ch}_n}(E,\nabla_0). \tag{8}$$

This defines $\widehat{\mathrm{ch}_n}(E,\nabla_1)$ and similarly for any t. By well-known formulas we obtain an expression for $\widehat{c_n}(E,\nabla_t)$ also from $\widehat{\mathrm{ch}_n}(E,\nabla_t)$. □

The referee suggested to use the variational formula to define the Chern–Simons class for any flat superconnection which belongs to a family where one member preserves the grading.

Corollary 3.8. *With notations as above, suppose $\{\nabla_t\}_t$ is a family of superconnections on E, such that ∇_0 preserves the grading. Assume that ∇_{t_0} is flat, for some t_0. There there is a uniquely determined class*

$$\widehat{c_n}(E, D_{t_0}) \in H^{2n-1}(M,\mathbb{R}/\mathbb{Z}),$$

for $n \geq 2$.

Proof. We use definition of $\widehat{c_n}(E,,D_{t_0}) \in \widehat{H^{2n}}_{\mathbb{R}}(M)$ from Lemma 3.7. Since D_{t_0} is flat, the Chern form is zero and the Character diagram (4), gives a Chern-Simons class. □

Corollary 3.9. *With notations as in Proposition 3.5, suppose $\nabla = D + L$ is flat superconnection on E, such that D preserves the grading. There there is a uniquely determined class*

$$\widehat{c_n}(E, D+L) \in H^{2n-1}(M,\mathbb{R}/\mathbb{Z}),$$

for $n \geq 2$.

Proof. Write a family of superconnections $\nabla_t := D + t.L$ on E, for $t \geq 0$. Now apply Corollary 3.8 directly to obtain the claim. □

We can extend the question of Cheeger and Simons as follows:

Question 3.10. *Suppose M is a supermanifold and (E,∇) is a complex flat superconnection on M such that its Chern-Simons classes are defined. Are the classes*

$$\widehat{c_n}(E,\nabla) \in H^{2n-1}(M,\mathbb{C}/\mathbb{Z})$$

torsion, if $n \geq 2$.

We will see some special situations in the next subsection where this question has a positive answer.

3.3. **Chern Simons classes for a morphism between flat connections.** Consider a homomorphism $u : E_0 \to E_1$ between complex vector bundles on a smooth manifold M. Then u determines a class in the K-group $K(M)$.

Let

$$L = i \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}. \tag{9}$$

Here u^* is the adjoint of u relative to a given metric (see [19]). Regard $E = E_0 \oplus E_1$ as a complex supervector bundle on M and $D_0 = D^0 + D^1$ be a superconnection on E preserving the grading. Then L is an odd degree endomorphism of E and as shown in [19], $D + L$ is a superconnection and its curvature form $F = (D + L)^2$ is an even form with values in End$E$.

Lemma 3.11. *Suppose (E_0, D^0) and (E_1, D^1) are flat connections and u and L are as above. Then we can define the Chern-Simons classes of the morphism u (which need not be a flat morphism) in the $\mathbb{C}/\mathbb{Z}$-cohomology of M by setting*

$$\widehat{c_n}(u) := \widehat{c_n}(E, D_0 + L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z}) \tag{10}$$

for $n \geq 1$, where $\widehat{c_n}(E, D_0 + L)$ are defined in Lemma 3.5.

Proof. The assumptions of Proposition 3.5 are fulfilled and we obtain uniquely defined classes

$$\widehat{c_n}(u) := \widehat{c_n}(E, D_0 + L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$$

for $n \geq 1$. □

We look at the following superconnection of the type $D + L$, considered by Quillen.

Theorem 3.12. (Relative Reznikov theorem) *Suppose $u : E_0 \to E_1$ is a (not necessarily flat) homomorphism between flat complex vector bundles (E_0, D^0) and (E_1, D^1) on a compact Kähler manifold M or a smooth complex quasi–projective variety M. Then the classes*

$$\widehat{c_i}(u) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Q})$$

are zero, for $i \geq 2$.

Proof. By Proposition 3.5, Remark 3.6 and Lemma 3.11 we have the explicit expression of the class

$$\widehat{c_n}(u) = \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1).$$

When M is a compact Kähler manifold then Reznikov's theorem [22] says that

$$\widehat{c_n}(E_0, D^0),\, \widehat{c_n}(E_1, D^1) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$$

are torsion, if $n \geq 2$. A similar result is true if M is a smooth complex quasi–projective variety, by [11]. Since the classes $\widehat{s_q}$ are expressed in terms of $\widehat{c_i}$ for $i \leq q$, the assertion follows. This proves the theorem □

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References

[1] F.A.Berezin, D.A.Leites, *Supermanifolds*, Soviet Math. Dokl. **16** (1975), 1218-1222.

[2] F.A. Berezin, G.I. Kac, *Lie groups with commuting and anticommuting parameters.* (Russian) Mat. Sb. (N.S.) **82** (124) 1970 343–359.

[3] Berezin, F. A. *Representations of the supergroup $U(p, q)$.* (Russian) Funkcional. Anal. i Priložen. 10 (1976), no. **3**, 70–71.

[4] F.A. Beresin, V.S. Retakh, *A method of computing characteristic classes of vector bundles.* Rep. Math. Phys. 18 (1980), no. **3**, 363–378.

[5] F.A. Berezin, *Introduction to superanalysis.* Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by Dimitri Leĭtes. Mathematical Physics and Applied Mathematics, **9**. D. Reidel Publishing Co., Dordrecht, 1987

[6] J. Cheeger, J. Simons, *Differential characters and geometric invariants*, Geometry and topology (College Park, Md., 1983/84), 50–80, Lecture Notes in Math., **1167**, Springer, Berlin, 1985.

[7] S.S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974), 48–69.

[8] P. Deligne, J.H. Morgan, *Notes on supersymmetry* (following Joseph Bernstein). Quantum fields and strings: a course for mathematicians, Vol. **1**, **2** (Princeton, NJ, 1996/1997), 41–97, Amer. Math. Soc., Providence, RI, 1999.

[9] J. Dupont, R. Hain, S. Zucker, *Regulators and characteristic classes of flat bundles*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 47–92, CRM Proc. Lecture Notes, **24**, Amer. Math. Soc., Providence, RI, 2000.

[10] H. Esnault, *Characteristic classes of flat bundles*, Topology 27 (1988), no. **3**, 323–352.

[11] J.N. Iyer, C.T. Simpson, *Regulators of canonical extensions are torsion; the smooth divisor case*, preprint 2007, arXiv math.AG/07070372.

[12] V.G. Kac, *Characters of typical representations of classical Lie superalgebras.* Comm. Algebra 5 (1977), no. **8**, 889–897.

[13] V.G. Kac, *Lie superalgebras.* Advances in Math. 26 (1977), no. **1**, 8–96.

[14] C. Laurent-Gengoux, *Secondary characteristic classes of super-foliations.* C. R. Math. Acad. Sci. Paris 339 (2004), no. **8**, 567–572.

[15] D.A. Leites, *Introduction to the theory of supermanifolds.* (Russian) Uspekhi Mat. Nauk 35 (1980), no. **1** (211), 3–57, 255. We used the yet unpublished book *Seminar on Supersymmetries* (English) by the same author.

[16] Y.I. Manin, *Gauge field theory and complex geometry.* Translated from the Russian by N. Koblitz and J. R. King. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **289**. Springer-Verlag, Berlin, 1988. x+297 pp.

[17] M. S. Narasimhan, S. Ramanan, *Existence of universal connections*, Amer. J. Math. **83** 1961 563–572.

[18] M. S. Narasimhan, S. Ramanan, *Existence of universal connections. II*, Amer. J. Math. **85** 1963 223–231.

[19] D. Quillen, *Superconnections and the Chern character.* Topology 24 (1985), no. **1**, 89–95.

[20] D. Quillen, *Superconnection character forms and the Cayley transform.* Topology 27 (1988), no. **2**, 211–238.

[21] A. Reznikov, *Rationality of Chern-Simons classes*, J. Differential Geom. 43 (1996), no. **3**, 674–692.

[22] A. Reznikov, *All regulators of flat bundles are torsion*, Ann. of Math. (2) 141 (1995), no. **2**, 373–386.

[23] A. Sergeev, *The invariant polynomials on simple Lie superalgebras.* Represent. Theory **3** (1999), 250–280

[24] V. Shander *Invariant functions on supermatrices*, arXiv:math/9810112.

[25] J. Simons, D. Sullivan, *Axiomatic Characterization of Ordinary Differential Cohomology*, arXiv math.AT/0701077.

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