CHERN-SIMONS CLASSES OF FLAT CONNECTIONS ON SUPERMANIFOLDS

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Abstract. In this note we define Chern-Simons classes of a superconnection $D+L$ on a complex supervector bundle E such that D is flat and preserves the grading, and L is an odd endomorphism of E on a supermanifold. As an application we obtain a definition of Chern-Simons classes of a (not necessarily flat) morphism between flat vector bundles on a smooth manifold. We extend Reznikov's theorem on triviality of these classes when the manifold is a compact Kähler manifold or a smooth complex quasi–projective variety, in degrees >1.

1. Introduction

Suppose $(X, \mathcal{C}^\infty_X)$ is a $\mathcal{C}^\infty$-differentiable manifold endowed with the structure sheaf $\mathcal{C}^\infty_X$ of smooth functions. Let E be a complex $\mathcal{C}^\infty$ vector bundle on X of rank r and equipped with a connection ∇. The Chern-Weil theory defines the Chern classes

$$c_i(E,\nabla) \in H^{2i}_{dR}(X,\mathbb{C}), \text{ for } i=0,1,...,r$$

in the de Rham cohomology of X. These classes are expressed in terms of the GL_r-invariant polynomials evaluated on the curvature form ∇^2.

Suppose E has a flat connection, i.e., $\nabla^2=0$. Then the de Rham Chern classes are zero. It is significant to define Chern-Simons classes for a flat connection. These are classes in the $\mathbb{C}/\mathbb{Z}$-cohomology and were defined by Chern-Cheeger-Simons in [6], [7].

Quillen has pointed out in [19],[20], a homomorphism $u: E_0 \to E_1$ between vector bundles on a smooth manifold M and inducing an isomorphism over a subset $A \subset M$ corresponds to an element in the relative K-group $K(M,A)$. A Chern character in the de Rham cohomology of M associated to the homomorphism u is computed in [19] whose class is shown to be equal to the difference $\mathrm{ch}(E_0)-\mathrm{ch}(E_1)$ of the Chern characters. This describes the Chern character of the homomorphism u. In fact, we think that it would be good to look at a quiver, i.e., a sequence of homomorphisms between vector bundles

$$E_0 \to E_1 \to ... \to E_r$$

over a smooth manifold and define the Chern character of the sequence in the de Rham cohomology. This will involve a study of $\mathbb{Z}_{r+1}$-graded objects, which we will look in the future. Quillens proof involves regarding $E=E_0\oplus E_1$ as a supervector bundle on M and

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D be any connection preserving the grading and associating an odd endomorphism of E, with respect to u and a choice of a metric.

In this paper, we want to look at a morphism u between flat vector bundles and extend Quillen's construction and define Chern-Simons classes for the morphism u. Hence it is relevant to define Chern-Simons classes for flat connections in the setting of supermanifolds, in a more general set-up.

For the definition of *supermanifolds*, due to F.A.Berezin and D. Leites, see [8] (as well as [1], [15]). The Chern classes of supervector bundles are defined in [4], on a supermanifold in the integral cohomology. We note that the usual Chern-Weil theory on smooth manifolds expresses de Rham Chern classes in terms of GL_n-invariant polynomials on the curvature form of a connection on a smooth vector bundle. In the supersetting, a study of the invariant polynomials has been carried out by Sergeev [23], following works by Berezin [3],[5] and Kac [12]. We do not know if the differential forms defined using the invariant polynomials of Sergeev give the de Rham Chern class of a super vectorbundle, as obtained by Quillen. In this paper we consider the Chern character defined by Quillen to define the Chern-Simons classes.

Let $(M, \mathcal{O}_M)$ denote a complex supermanifold and $(M_B, \mathcal{C}^\infty_M)$ denote the underlying $\mathcal{C}^\infty$-manifold.

With notations as in [8] or §2, we show

Theorem 1.1. *Suppose $(M, \mathcal{O}_M)$ is a complex supermanifold. Let $\mathcal{E}^{r|s}$ be a complex supervector bundle on $(M, \mathcal{O}_M)$ equipped with a superconnection $\nabla = D + L$ such that D preserves the grading and is flat, and L is an odd endomorphism of $\mathcal{E}^{r|s}$. Then there exists uniquely determined Chern-Simons classes*

$$\widehat{c_i}(\mathcal{E}^{r|s}, \nabla) \in H^{2i-1}(M_B, \mathbb{C}/\mathbb{Z})$$

for $i > 0$. Furthermore, if M_B is a compact Kähler manifold or a smooth complex quasi–projective variety, then these classes are torsion, in degrees > 1.

This can be thought of as an extension of Reznikov's fundamental theorem [22] on rationality of Chern-Simons classes on compact Kähler manifold, in the setting of supermanifolds. We also define Chern-Simons classes of a (not necessarily flat) homomorphism $u : E_0 \to E_1$ between flat complex vector bundles, extending Quillen's construction of the de Rham Chern character. Then we prove a relative Reznikov theorem (see Theorem 3.8) for the classes of the morphism u.

2. Preliminaries

We briefly recall the definitions and terminologies from [15] and from the notes by Deligne and Morgan [8].

Let $\mathcal{C}^\infty$ be the sheaf of C^∞-functions on $\mathbb{R}^p$. The space $\mathbb{R}^{p|q}$ is the topological space $\mathbb{R}^p$, endowed with the sheaf $\mathcal{C}^\infty[\theta_1, ..., \theta_q]$ of supercommutative super $\mathbb{R}$-algebras, freely generated over $\mathcal{C}^\infty$ by the anticommuting $\theta_1, ..., \theta_q$. The coordinates t^i of $\mathbb{R}^p$ and the θ_j and all generators of $\mathcal{C}^\infty$ obtained from them by any automorphism are said to be the coordinates of $\mathbb{R}^{p|q}$. A supermanifold M of dimension $p|q$ is a topological space M_B (or also called as the body manifold with the structure sheaf $\mathcal{C}^\infty_M$) endowed with a sheaf of super $\mathbb{R}$-algebras which is locally isomorphic to $\mathbb{R}^{p|q}$. The structure sheaf of M is denoted by $\mathcal{O}_M$. We denote $p|q$, the real dimension of the supermanifold M.

On $M = \mathbb{R}^{p|q}$, the even derivations $\partial/\partial t^i$ and the odd derivations $\partial/\partial\theta^j$ are defined.

Proposition 2.1. [15, 2.2.3] *The $\mathcal{O}_M$-module of $\mathbb{R}$-linear derivations of $\mathcal{O}_M$ is free of dimension $p|q$, with basis: the $\partial/\partial t^i$ and the $\partial/\partial\theta^j$.*

Complex supermanifolds are topological spaces endowed with a sheaf of super $\mathbb{C}$-algebras, locally isomorphic to some $(\mathbb{C}^p, \mathcal{O}[\theta^1, ..., \theta^q])$. Here $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^p$. As before we denote $p|q$, the complex dimension of the complex supermanifold M.

Suppose R be a commutative superalgebra and the standard free module $A^{r|s}$ is the module freely generated by even elements $e_1, ..., e_r$ and odd elements $f_1, ..., f_s$. An automorphism of $A^{r|s}$ is represented by an invertible matrix

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \tag{1}$$

such that the $(r\times r)$–matrix X_1 and the $(s\times s)$–matrix X_4 have even entries and the $(s\times r)$–matrix X_3 and the $(r \times s)$–matrix X_2 have odd entries. The group of all automorphisms of $A^{r|s}$ is denoted by $GL(r, s)$.

Suppose M is a supermanifold and locally it looks like $\mathbb{R}^{p|q}$ as above. A *complex supervector bundle* V on M is a fiber bundle V over M with typical fiber $\mathbb{C}^{r|s}$ and structural group $GL(r, s)$. Alternately, it can be considered as a sheaf of $\mathcal{O}_M$-supermodules $\mathcal{V}$, locally free of rank $r|s$.

The tangent bundle $\mathcal{T}_M$ is the $\mathcal{O}_M$-module of derivations of $\mathcal{O}_M$ and is a supervector bundle of rank $p|q$. The cotangent bundle Ω^1_M is the dual of $\mathcal{T}_M$. There is a differential $d : \mathcal{O}_M \longrightarrow \Omega^1_M$, giving rise to the super de Rham complex $\Omega^\bullet_M$ on M.

Lemma 2.2. (Poincaré lemma)[8, p.73] *The complex $\Omega^\bullet_M$ is a resolution of the constant sheaf on the body manifold M_B.*

In particular, the cohomology of M_B can be computed by the super de Rham complex:

$$H^*(M_B, \mathbb{R}) = H^*(\Gamma(M, \Omega^\bullet_M)).$$

We briefly review the group of differential characters and Chern-Simons classes on a smooth manifold X.

2.1. **Analytic differential characters on** X [6]**.** Let $S_k(X)$ denote the group of k-dimensional smooth singular chains on X, with integer coefficients. Let $Z_k(X)$ denote the subgroup of cycles. Let us denote

$$S^\bullet(X,\mathbb{Z}) := \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(X),\mathbb{Z})$$

the complex of $\mathbb{Z}$ -valued smooth singular cochains, whose boundary operator is denoted by δ. The group of smooth differential k-forms on X with complex coefficients is denoted by $A^k(X)$ and the subgroup of closed forms by $A^k_{cl}(X)$. Then $A^\bullet(X)$ is canonically embedded in $S^\bullet(X)$, by integrating forms against the smooth singular chains. In fact, we have an embedding

$$i_{\mathbb{Z}} : A^\bullet(X) \hookrightarrow S^\bullet(X,\mathbb{C}/\mathbb{Z}).$$

The group of differential characters of degree k is defined as

$$\widehat{H^k}_{\mathbb{C}}(X) := \{(f,\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(X),\mathbb{C}/\mathbb{Z}) \oplus A^k(X) : \delta(f) = i_{\mathbb{Z}}(\alpha) \text{ and } d\alpha = 0\}.$$

There is a canonical and functorial exact sequence:

$$0 \longrightarrow H^{k-1}(X,\mathbb{C}/\mathbb{Z}) \longrightarrow \widehat{H^k}_{\mathbb{C}}(X) \longrightarrow A^k_{\mathbb{Z}}(X) \longrightarrow 0. \tag{2}$$

Here $A^k_{\mathbb{Z}}(X) := \ker(A^k_{cl}(X) \longrightarrow H^k(X,\mathbb{C}/\mathbb{Z}))$.

Similarly, one can define the group of differential characters $\widehat{H^k}_{\mathbb{R}}(X)$ which have $\mathbb{R}/\mathbb{Z}$-coefficients.

2.2. **Cheeger-Chern-Simons classes.** Suppose (E,θ) is a vector bundle with a connection on X. Then the characteristic forms

$$c_k(E,\theta) \in A^{2k}_{cl}(X,\mathbb{Z})$$

for $0 \leq k \leq r$=rank (E), are defined using the universal Weil homomorphism [7].

The characteristic classes

$$\widehat{c_k}(E,\theta) \in \widehat{H^{2k}}_{\mathbb{C}}(X)$$

are defined in [6] using a factorization of the universal Weil homomorphism. These classes are functorial lifting of the forms $c_k(E,\theta)$.

Similarly, there are classes

$$\widehat{c_k}(E,\theta) \in \widehat{H^{2k}}_{\mathbb{R}}(X).$$

Remark 2.3. *If the forms $c_k(E,\theta)$ are zero, then the classes $\widehat{c_k}(E,\theta)$ lie in the cohomology $H^{2k-1}(X,\mathbb{C}/\mathbb{Z})$. If (E,θ) is a flat bundle, then $c_k(E,\theta)=0$ and the classes $\widehat{c_k}(E,\theta)$ are called as the Chern-Simons classes of (E,θ). Notice that the class depends on the choice of θ.*

Beilinson's theory of universal Chern-Simons classes yield classes for a flat connection (E,θ),

$$\widehat{c_k}(E,\theta) \in H^{2k-1}(X,\mathbb{C}/\mathbb{Z})$$

which are functorial and additive over exact sequences (see [9] and [10] for another construction).

3. Chern-Simons classes of flat connections on supermanifolds

Let $(M, \mathcal{O}_M)$ be a complex supermanifold of dimension $p|q$. Consider the sheaf of differentials Ω^1_M on M and let $\mathcal{E}^{r|s}$ be a complex supervector bundle on M of rank $r|s$.

Lemma 3.1. *Given a complex supervector bundle $\mathcal{E}^{r|s}$ of rank $r|s$ on M, there exists a direct sum decomposition*

$$E = E_0 \oplus E_1$$

for some complex smooth vector bundles E_0 and E_1 of rank r and rank s respectively, on the underlying $\mathcal{C}^\infty$-manifold M_B.

Proof. A rank $r|s$ complex supervector bundle $\mathcal{E}^{r|s}$ determines two complex smooth bundles E_0 and E_1 on the underlying smooth manifold M_B of M as follows. One considers the body map

$$\mathcal{O}_M \longrightarrow \mathcal{C}^\infty_M \otimes \mathbb{C}$$

which is obtained by forgetting the local anticommuting variables θ_j. Let $\overline{\mathcal{E}^{r|s}}$ denote the sheaf of (super)sections of $\mathcal{E}^{r|s}$. Then $\overline{E_{r+s}} := \overline{\mathcal{E}^{r|s}} \otimes_{\mathcal{O}_M} (\mathcal{C}^\infty_M \otimes \mathbb{C})$ is the sheaf of sections of a rank $r+s$ smooth complex vector bundle E_{r+s} on the body manifold M_B. Locally, the sheaf $\mathcal{E}^{r|s}$ is generated by r even elements and s odd elements as a $\mathcal{O}_M = \mathcal{C}^\infty_M[\theta_1, ..., \theta_q]$–module. Hence tensoring with $\mathcal{C}^\infty_M$ locally gives a rank $r+s$ free $\mathcal{C}^\infty_M$–module given by the same generators. This implies that the complex vector bundle E_{r+r} is of rank $r+s$. Now, we notice that the structural group of $\mathcal{E}^{r|s}$ is $GL(r,s)$ and the structural group of the vector bundle E_{r+s} factors via the projection

$$GL(r,s) \to GL(r+s).$$

Using the description of the elements in $GL(r,s)$ in (1), it follows that the image under this projection consists of block diagonal matrices of sizes $r \times r$ and $s \times s$.

This implies that the matrix of the transition functions of E_{r+s} is a block diagonal matrix of rank r and rank s which correspond to smooth complex vector bundles E_0 and E_1 such that $r = \text{rank } E_0$ and $s = \text{rank } E_1$ on M_B. □

In view of the above lemma, we may regard a supervector bundle $\mathcal{E}^{r|s}$ on M as a supervector bundle $E = E_0 \oplus E_1$, on the underlying $\mathcal{C}^\infty$-manifold M_B where E_0 and E_1 are $\mathcal{C}^\infty$-vector bundles on M_B.

3.1. **Superconnections.** Let $\mathcal{E}^{r|s} = E = E_0 \oplus E_1$ be a complex supervector bundle on a manifold M_B. Let $\Omega(M_B) = \oplus \Omega^p(M_B)$ be the algebra of smooth differential forms with complex coefficients. Let

$$\Omega(M_B, E) := \Omega(M_B) \otimes_{\Omega^0(M_B)} \Omega^0(M_B, E).$$

where $\Omega^0(M_B, E)$ is the space of (super)sections of E.

Then $\Omega(M_B, E)$ has a grading with respect to $\mathbb{Z} \times \mathbb{Z}_2$

A *superconnection* D on $\mathcal{E}^{r|s}$ is an operator on $\Omega(M_B, E)$ of odd degree satisfying the derivation property

$$D(\omega\alpha) = (d\omega)\alpha + (-1)^{deg\,\omega}\nabla\alpha.$$

For example, a connection on E preserving the grading when extended to an operator on $\Omega(M_B, E)$ in the usual way determines a superconnection.

The *curvature* of a superconnection D is the even degree operator $D^2 := D \circ D$ on $\Omega(M_B, E)$.

A superconnection is said to be *flat* if $D^2 = 0$. We call the pair $(\mathcal{E}^{r|s}, D)$ as a flat complex supervector bundle.

We want to define Chern-Simons classes of (E, D) when D is a flat superconnection. It is not immediately clear that D induces a flat connection on the individual bundle E_0 and E_1.

For this purpose, we look at the situation, considered by Quillen [19].

3.2. **Quillen's construction.** Suppose M is a supermanifold and E is a complex supervector bundle on M. Regard $E = E_0 \oplus E_1$ as a complex supervector bundle on M in view of Lemma 3.1, where E_0 and E_1 are smooth vector bundles on the body manifold M_B. Under this identification we omit the suffix B from M_B and without confusion we write $M = M_B$ in the following discussion.

Suppose E is equipped with a superconnection D. From the curvature D^2, one can construct differential forms

$$\mathrm{str}(D^2)^n = \mathrm{str}D^{2n}$$

in $\Omega(M)^{even}$. These are even forms since the supertrace preserves the grading.

We have,

Theorem 3.2. *The form* $\mathrm{strD^{2n}}$ *is closed, and its de Rham cohomology class is independent of the choice of superconnection D.*

Proof. See [19, Theorem, p.91]. □

Quillen described the (super) Chern character of E in terms of the usual Chern characters of E_0 and E_1 in the following situation.

Regard $E = E_0 \oplus E_1$ as a complex supervector bundle and $D_0 = D^0 + D^1$ be a connection on E preserving the grading. Let L be an odd degree endomorphism of E and write $D_t := D + t.L$ where t is a parameter.

Proposition 3.3. (Quillen)[19] *Replacing L by $t.L$, where t is a parameter, one obtains a family of forms*

$$str\, e^{(D+tL)^2} = str\, e^{r^2L^2+t[D,L]+D^2} \tag{3}$$

all of which represent the Chern character $ch(E_0) - ch(E_1)$ in the de Rham cohomology of M. Here str denotes the supertrace.

□

The referee has pointed out that the above computations on a supermanifold produces pseudodifferential forms. For our purpose, it suffices to note that the trace form in (3) defines a closed differential form whose de Rham class is independent of the superconnection [19, Theorem p.91].

In this situation we want to define uniquely determined Chern-Simons classes of (E, D_t) which is independent of t and if D^0, D^1 are flat connections. For this purpose, we look at the *Character diagram* of Simons and Sullivan (see [25]):

$$\begin{tikzcd}
0 \arrow[drr] & & & & & & 0 \\
\arrow[dr] & & H^{k-1}(\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "i_1"] & & H^k(\mathbb{Z}) \arrow[dr, "r"] \arrow[urr] & & \\
& H^{k-1}(\mathbb{R}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & \widehat{H^k}_{\mathbb{R}}(M) \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^k(\mathbb{R}) \arrow[ur] \arrow[dr] & \\
\arrow[ur] & & A^{k-1}/A^{k-1}_{\mathbb{Z}} \arrow[rr, "d"] \arrow[ur, "i_2"] & & A^k_{\mathbb{Z}} \arrow[ur, "s"] \arrow[drr] & & \\
0 \arrow[urr] & & & & & & 0
\end{tikzcd} \tag{4}$$

The diagonal sequences are exact and (α, B, r) is the Bockstein long exact sequence associated to the coefficient sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Also (β, d, s) is another long exact sequence in which β and s are defined via the de Rham theorem. (A similar diagram holds by replacing $\mathbb{R}$ with $\mathbb{C}$).

Lemma 3.4. *Suppose (F, ∇) is a smooth flat connection on a manifold M. Then the Chern class $c_i(F) \in H^{2i}(M, \mathbb{Z})$ vanishes in $H^{2i}(M, \mathbb{R})$. There is a unique lifting $\widehat{c_i}(F, \nabla) \in H^{2i-1}(M, \mathbb{R}/\mathbb{Z})$ of the integral class $c_i(F)$, for $i > 0$.*

Proof. The first assertion is by the Chern-Weil theory. The second assertion is by the Chern-Simons-Cheeger construction [6]. □

Consider the total Chern class

$$c(F) = 1 + c_1(F) + ... + c_f(F)$$

and the total Segre class

$$s(F) = 1 + s_1(F) + ... + s_f(F).$$

Then we have the relations

$$s(F) = \frac{1}{c(F)} \tag{5}$$

and

$$c(F - G) = c(F).s(G) \tag{6}$$

where G is any other vector bundle. These relations also hold if we replace the classes $c_i(F)$ by $\widehat{c_i}(F, \nabla)$ and $s_i(F)$ by $\widehat{s_i}(F, \nabla)$ which are defined by the relation (5). See [6, p.64-65].

To define a canonical lifting in $\widehat{H^*_{\mathbb{R}}}(M)$ of the trace form (3) associated to the superconnection (E, D_t), one would need a notion of universal superconnection analogous to the universal connections defined by Narasimhan and Ramanan [17], [18], which we may look in the future.

Nonetheless, we consider the family of superconnection $D_t = D + t.L$ as above. Consider the product manifold $\mathbb{R} \times M$ and the pullback $pr_2^* E$ of the bundle E. This bundle is equipped with a superconnection

$$\bar{D} := dt\partial_t + D'$$

whose restriction to $\{t\} \times M$ is D_t. In terms of local trivialization of $E = M \times V$ we can describe $\bar{D}$, D' as follows. Write $D_t = d_M + \theta_t$, where θ_t is a family of one-forms on M with values in End$V$ and let θ be the form on $\mathbb{R} \times M$ not involving dt and having the restriction θ_t on $\{t\} \times M$. Then $D' = d_M + \theta$ and

$$\bar{D} = dt\partial_t + D' = (dt\partial_t + d_M) + \theta = d_{\mathbb{R}\times M} + \theta.$$

See also [19, p.91].

By the homotopy property of de Rham cohomology, it follows that the class of strD_t^{2n} in $H^{2n}(M, \mathbb{R})$ is independent of t.

Proposition 3.5. *Suppose the superconnection $D_0 = D^0 \oplus D^1$ on the supervector bundle $E = E_0 \oplus E_1$ preserves the grading and the individual smooth connections D^0 and D^1 are flat connections on E_0 and E_1 respectively. Then $D_t = D_0 + t.L$ is a superconnection* *on E where L is an odd endomorphism of E. Then there is a uniquely determined class $\widehat{c_n}(E, D_t) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$, independent of t. Moreover this class is equal to*

$$\widehat{c_n}(E, D_0 + L) = \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1)$$

Proof. We notice that the trace forms are integral valued. This implies that the Chern class associated to these forms lies in the integral cohomology $H^{2n}(M, \mathbb{Z})$ which is independent of t in $H^{2n}(M, \mathbb{R})$, by Quillen's Theorem 3.2. This determines a class in $H^{2n}(M, \mathbb{Z})$ independent of t. But this class vanishes in $H^{2n}(M, \mathbb{R})$ since D_0 has components D^0 and D^1 which are flat connections, hence $D_0^2 = 0$. Using the Bockstein operator in (4), we conclude that there is a class

$$\widehat{c_n}(E, D_t) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$$

which is independent of t and we denote this class by $\widehat{c_n}(E, D_0 + L)$. We get a uniquely determined class $\widehat{c_n}(E, D_0 + L) = \widehat{c_n}(E, D_0) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$, as follows.

To get an expression for this class, we notice that by Quillen's result Proposition 3.3 the (super) Chern character form of E is the difference $\mathrm{ch}(E_0) - \mathrm{ch}(E_1)$ in the integral cohomology. In particular we want to lift the integral Chern classes of $E_0 - E_1$ in the $\mathbb{R}/\mathbb{Z}$-cohomology. The relations in (5) and (6) give the formula

$$\widehat{c_n}(E, D_0 + L) := \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1).$$

The uniqueness of $\widehat{c_n}(E, D_0 + L)$ follows from the uniqueness of the Chern-Simons classes $\widehat{c_p}(E_0, D^0)$ and $\widehat{c_q}(E_1, D^1)$, see Lemma 3.4. This concludes the lemma. □

Remark 3.6. *All the above constructions follow verbatim by replacing $\mathbb{R}/\mathbb{Z}$-coefficients with $\mathbb{C}/\mathbb{Z}$-coefficients. We call the resulting classes $\widehat{c_n}(E, D_0 + L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$ as the Chern-Simons classes of the superconnection $D_0 + L$ (or $D_0 + t.L$ for a parameter t).*

3.3. **Chern Simons classes for a morphism between flat connections.** Consider a homomorphism $u : E_0 \to E_1$ between complex vector bundles on a smooth manifold M. Then u determines a class in the K-group $K(M)$.

Let

$$L = i \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}. \tag{7}$$

Here u^* is the adjoint of u relative to a given metric (see [19]). Regard $E = E_0 \oplus E_1$ as a complex supervector bundle on M and $D_0 = D^0 + D^1$ be a superconnection on E preserving the grading. Then L is an odd degree endomorphism of E and as shown in [19], $D + L$ is a superconnection and its curvature form $F = (D + L)^2$ is an even form with values in $\mathrm{End}E$.

Lemma 3.7. *Suppose (E_0, D^0) and (E_1, D^1) are flat connections and u and L are as above. Then we can define the Chern-Simons classes of the morphism u (which need not be a flat morphism) in the $\mathbb{C}/\mathbb{Z}$-cohomology of M by setting*

$$\widehat{c_n}(u) := \widehat{c_n}(E, D_0 + L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z}) \tag{8}$$

for $n \geq 1$, where $\widehat{c_n}(E, D_0 + L)$ are defined in Lemma 3.5.

Proof. The assumptions of Proposition 3.5 are fulfilled and we obtain uniquely defined classes

$$\widehat{c_n}(u) := \widehat{c_n}(E, D_0 + L) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$$

for $n \geq 1$. □

Theorem 3.8. (Relative Reznikov theorem) *Suppose $u : E_0 \to E_1$ is a (not necessarily flat) homomorphism between flat complex vector bundles (E_0, D^0) and (E_1, D^1) on a compact Kähler manifold M or a smooth complex quasi–projective variety M. Then the classes*

$$\widehat{c_i}(u) \in H^{2i-1}(M, \mathbb{C}/\mathbb{Q})$$

are zero, for $i \geq 2$.

Proof. By Proposition 3.5, Remark 3.6 and Lemma 3.7 we have the explicit expression of the class

$$\widehat{c_n}(u) = \sum_{p+q=n} \widehat{c_p}(E_0, D^0).\widehat{s_q}(E_1, D^1).$$

When M is a compact Kähler manifold then Reznikov's theorem [22] says that

$$\widehat{c_n}(E_0, D^0),\, \widehat{c_n}(E_1, D^1) \in H^{2n-1}(M, \mathbb{C}/\mathbb{Z})$$

are torsion, if $n \geq 2$. A similar result is true if M is a smooth complex quasi–projective variety, by [11]. Since the classes $\widehat{s_q}$ are expressed in terms of $\widehat{c_i}$ for $i \leq q$, the assertion follows. This proves the theorem □

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