CHERN-SIMONS CLASSES FOR A SUPERCONNECTION

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ABSTRACT. In this note we define the Chern-Simons classes of a flat superconnection, $D+L$, on a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E on a manifold such that D preserves the grading and L is an odd endomorphism of E. As an application, we obtain a definition of Chern-Simons classes of a (not necessarily flat) morphism between flat vector bundles on a smooth manifold. An application of Reznikov's theorem shows the triviality of these classes when the manifold is a compact Kähler manifold or a smooth complex quasi–projective variety in degrees > 1.

1. INTRODUCTION

Suppose $(M, \mathcal{C}^\infty_M)$ is a $\mathcal{C}^\infty$-differentiable manifold endowed with the structure sheaf $\mathcal{C}^\infty_M$ of smooth functions. Let E be a complex $\mathcal{C}^\infty$ vector bundle on M of rank r equipped with a connection ∇. The Chern-Weil theory defines the Chern classes

$$c_i(E,\nabla) \in H^{2i}_{dR}(M,\mathbb{R}), \text{ for } i = 0, 1, ..., r$$

in the de Rham cohomology of M. These classes are expressed in terms of the GL_r-invariant polynomials evaluated on the curvature form ∇^2.

Suppose E has a flat connection; i.e., $\nabla^2 = 0$. Then the de Rham Chern classes are zero. It is significant to define Chern-Simons classes for a flat connection. These are classes in the $\mathbb{R}/\mathbb{Z}$-cohomology and were defined by Chern-Cheeger-Simons in [7], [8].

In the supersetting a study of the $GL(r,s)$–invariant functions has been carried out by Berezin [5]. Quillen [21], [22] has looked into the case of defining the Chern character of a superconnection $D + L$ on a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle E. Here D is a smooth connection preserving the grading on E and L is an odd endomorphism of E. The differential forms defined by Quillen are obtained from the Chern character str e^{D+L} (here str denotes the supertrace, see (2) below) and have $\frac{1}{i!}\mathrm{str}((D+L)^{2i})$ as their term in degree $2i$. This term is obtained by substituting the curvature form of the superconnection in the $GL(r,s)$–invariant polynomial, $P_i := \mathrm{str}(A_1A_2...A_i)$ where A_j are supermatrices (see also Remark 3.6). Even though $e^{(D+L)^2}$ is a power series, each degree term is given by a polynomial.

To define differential characters which are unique liftings of the Chern forms (see §2.3), we need to look at the standard polynomials P_i. Notice that the forms $P_i((D + L)^{2i})$

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are polynomial expressions in the curvature form and the first Chern form is a constant multiple of $P_1((D+L)^2)$.

In this paper we use the standard polynomials P_i above to define the Chern-Simons classes.

With notations as in §2, we show:

Theorem 1.1. *Suppose $\{\nabla_t\}_t$ is a family of superconnections on a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E, such that ∇_0 preserves the $\mathbb{Z}/2\mathbb{Z}$-grading. Suppose ∇_{t_0} is flat for some t_0. Then there is a uniquely determined Chern-Simons class*

$$CS_k(E,\nabla_{t_0}) \in H^{2k-1}(M,\mathbb{R}/\mathbb{Z}),$$

for $k \geq 1$.

In particular this applies to the following situation:

Corollary 1.2. *Suppose M is a smooth manifold. Let E be a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle on M equipped with a superconnection $\nabla = D+L$ such that D preserves the $\mathbb{Z}/2\mathbb{Z}$-grading and L is an odd endomorphism of E. Assume that ∇ is a flat superconnection. Then there exist uniquely determined Chern-Simons classes*

$$CS_k(E,\nabla) \in H^{2k-1}(M,\mathbb{R}/\mathbb{Z})$$

for $k > 0$. Furthermore, if M is a compact Kähler manifold or a smooth complex quasi–projective variety and D itself is a flat smooth connection then $CS_k(E,\nabla) = 0$ in $H^{2k-1}(M,\mathbb{R}/\mathbb{Q})$ for $k > 1$.

The last claim is an application of Reznikov's theorem [24] on rationality of Chern-Simons classes on compact Kähler manifold.

A homomorphism $u : E_0 \to E_1$ between vector bundles on a smooth manifold M that induces an isomorphism over a subset $A \subset M$ corresponds to an element in the relative K-group $K(M,A)$. Quillen [21], [22] has shown using his definition of a Chern character how to obtain a de Rham image of this relative class in the relative de Rham cohomology. It may be possible to define a relative differential cohomology and define differential characters for u in the relative differential cohomology, when E_0 and E_1 are equipped with smooth connections and isomorphic over some open subset. This will require some more analysis and is not treated in this paper. However, in this paper, we define the Chern-Simons classes in the absolute $\mathbb{R}/\mathbb{Z}$-cohomology of M of a (not necessarily flat) homomorphism $u : E_0 \to E_1$ between flat complex vector bundles, extending Quillen's construction of the de Rham Chern character. It turns out that u does not play any role and one can apply Reznikov's theorem in this situation (see Theorem 3.12). More generally, the question of Cheeger-Simons on the rationality of these classes can be posed (see Question 3.10) for flat superconnections of the type $D+L$.

It is evident that the question of defining the Chern-Simons invariants for a general superconnection and the classes in other cohomology theories is interesting. Voronov and

Zorich ([28], [29]) have given an analog of the ordinary de Rham theory/ integration theory on manifolds in the context of supermanifolds. It is important to look at superconnections on complex supervector bundles which are not equivalent to a split supervector bundle $E = E_0 \oplus E_1$ and define their Chern-Simons invariants in the de Rham algebra of the supermanifold. This will involve developing a theory of superconnections, curvature, Chern-Simons invariants and differential characters using the super de Rham complex.

2. Preliminaries

2.1. **Supertrace of a matrix.** Suppose A is a commutative superalgebra. The standard free module $A^{r|s}$ is the module freely generated by even elements $e_1, ..., e_r$ and odd elements $f_1, ..., f_s$. An automorphism of $A^{r|s}$ is represented by an invertible matrix

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \tag{1}$$

such that the $(r\times r)$–matrix X_1 and the $(s\times s)$–matrix X_4 have even entries and the $(s\times r)$–matrix X_3 and the $(r \times s)$–matrix X_2 have odd entries. The group of all automorphisms of $A^{r|s}$ is denoted by $GL(r, s)$.

The *supertrace* of the matrix X is the difference of the usual trace of the matrices X_1 and X_4; that is,

$$\mathrm{str(X)} := \mathrm{tr(X_1)} - \mathrm{tr(X_4)} \tag{2}$$

2.2. **Analytic differential characters on** X **[7].** Suppose M is a smooth manifold. Let $S_k(M)$ denote the group of k-dimensional smooth singular chains on M with integer coefficients. Let $Z_k(M)$ denote the subgroup of cycles. For any abelian group Λ, let us denote

$$S^\bullet(M, \Lambda) := \mathrm{Hom}_{\mathbb{Z}}(S_\bullet(M), \Lambda)$$

the complex of Λ -valued smooth singular cochains with boundary operator δ. The group of smooth differential k-forms on M with real coefficients is denoted by $A^k(M)$ and the subgroup of closed forms by $A^k_{cl}(M)$. Then $A^\bullet(M)$ is canonically embedded in $S^\bullet(M, \mathbb{R})$, by integrating forms against the smooth singular chains. In fact, there is an embedding

$$i_{\mathbb{Z}} : A^\bullet(M) \hookrightarrow S^\bullet(M, \mathbb{R}/\mathbb{Z}).$$

The group of differential characters of degree k is defined as

$$\widehat{H^k}(M)_{\mathbb{R}} := \{(f, \alpha) \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \oplus A^k(M) : \delta(f) = i_{\mathbb{Z}}(\alpha) \text{ and } d\alpha = 0\}.$$

There is a canonical and functorial exact sequence:

$$0 \longrightarrow H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{H^k}(M)_{\mathbb{R}} \longrightarrow A^k_{\mathbb{Z}}(M) \longrightarrow 0. \tag{3}$$

Here $A^k_{\mathbb{Z}}(M) := \ker(A^k_{cl}(M) \longrightarrow H^k(M, \mathbb{R}/\mathbb{Z}))$.

Similarly, one can define the group of differential characters $\widehat{H^k}(M)_{\mathbb{C}}$ which have $\mathbb{C}/\mathbb{Z}$-coefficients.

2.3. **Cheeger-Chern-Simons classes.** Suppose (E,θ) is a complex vector bundle with a connection on M. Then the characteristic forms

$$c_k(E,\theta) \in A^{2k}_{cl}(M,\mathbb{Z})$$

for $0 \leq k \leq r$=rank (E) are defined using the universal Weil homomorphism [8].

The characteristic classes or the differential characters

$$\widehat{c_k}(E,\theta) \in \widehat{H^{2k}}(M)_{\mathbb{R}}$$

are defined in [7] using a factorization of the universal Weil homomorphism. These classes are functorial liftings of the forms $c_k(E,\theta)$.

Similarly, there are classes

$$\widehat{c_k}(E,\theta)_{\mathbb{C}} \in \widehat{H^{2k}}(M)_{\mathbb{C}}.$$

Remark 2.1. *If the forms $c_k(E,\theta)$ are zero then the classes $\widehat{c_k}(E,\theta)$ lie in the cohomology $H^{2k-1}(M,\mathbb{R}/\mathbb{Z})$. In particular, if (E,θ) is a flat bundle then $c_k(E,\theta)=0$. In this situation, the class $\widehat{c_k}(E,\theta)$ which lies in $H^{2k-1}(M,\mathbb{R}/\mathbb{Z})$ will be denoted by $CS_k(E,\theta)$. These are called the Chern-Cheeger-Simons classes or the Chern-Simons classes of (E,θ).*

3. Chern-Simons classes of flat superconnections

In this paper, a complex supervector bundle will be identified with the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E = E_0 \oplus E_1$. We recall the definitions of superconnection and curvature below for the bundle E and for the definition in general for superconnections on a complex supervector bundle, see [9, p.77].

3.1. **Superconnections.** Let $E = E_0 \oplus E_1$ be a complex supervector bundle on a manifold M. Let $A(M) = \oplus_{p\geq 0} A^p(M)$ be the algebra of smooth differential forms. Let

$$A(M,E) := A(M) \otimes_{A^0(M)} A^0(M,E).$$

Here $A^0(M,E) := A^0(M,E_0) \oplus A^0(M,E_1)$ forms a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector space. This is the space of supersections of E.

Then $A(M,E)$ has a grading with respect to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

A *superconnection* D on E is an operator on $A(M,E)$ of odd degree (with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading on $A^0(M,E)$) satisfying the derivation property

$$D(\omega\alpha) = (d\omega)\alpha + (-1)^{deg\,\omega} D\alpha.$$

For example, a $\mathbb{Z}/2\mathbb{Z}$-graded connection on E when extended to an operator on $A(M,E)$ in the usual way determines a superconnection.

In local coordinates when E is trivial, E looks like $M \times V$ where V is a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space. Here, a superconnection D is of the form $d+\theta$, where θ is an odd element of $A(M) \otimes \mathrm{End}(V)$.

The *curvature* of a superconnection D is the even degree operator $D^2 := D \circ D$ on $A(M,E)$.

A superconnection D is said to be *flat* if $D^2 = 0$. We then call the pair (E,D) a flat complex supervector bundle.

We want to define Chern-Simons classes of (E,D) when D is a flat superconnection. For this purpose we look at the situation considered by Quillen [21] when the superconnection is locally of the form $d+\theta$ where

$$\theta = A + L \in A^1(M) \otimes (\mathrm{End}\ V)^0 \oplus A^0(M) \otimes (\mathrm{End}\ V)^1.$$

It is an interesting question to define Chern-Simons classes for arbitrary flat superconnections $d+\theta$, where θ is an arbitrary odd element of $A(M) \otimes \mathrm{End}(V)$, which we do not know how to treat yet.

3.2. **Quillen's construction.** Suppose M is a manifold and $E = E_0 \oplus E_1$ is a complex supervector bundle on M where E_0 and E_1 are smooth vector bundles on M.

Suppose E is equipped with a superconnection D, as in the previous subsection. The curvature form D^2 allows one to construct differential forms

$$\mathrm{str}(D^2)^n = \mathrm{str} D^{2n}$$

in $A(M)^{even}$. These are even forms since the supertrace preserves the grading. We have,

Theorem 3.1. *The form strD^{2n} is closed and its de Rham cohomology class is independent of the choice of superconnection D.*

Proof. See [21, Theorem, p.91]. □

Quillen described the (super) Chern character of E in terms of the usual Chern characters of E_0 and E_1 in the following situation:

Regard $E = E_0 \oplus E_1$ as a complex supervector bundle. Let $D = D_0 + D_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded connection on E, and L be an odd degree endomorphism of E. Set $\nabla_t := D + t.L$ where t is a parameter.

Proposition 3.2. (Quillen)[21] *Replacing L by tL, where t is a parameter, one obtains a family of forms*

$$\mathrm{str}\, \mathrm{e}^{(\mathrm{D}+\mathrm{tL})^2} = \mathrm{str}\, \mathrm{e}^{\mathrm{t}^2\mathrm{L}^2 + \mathrm{t}[\mathrm{D},\mathrm{L}] + \mathrm{D}^2} \tag{4}$$

all of which represent the Chern character $ch(E_0) - ch(E_1)$ in the de Rham cohomology of M. Here str *denotes the supertrace.*

□

The above computations produce pseudodifferential forms (see [6] for the definition). For our purpose, since we are looking for the topological invariants associated to the split bundle $E_0 \oplus E_1$, it suffices to note that the trace form in (4) defines a closed differential form whose de Rham class is independent of the superconnection [21, Theorem p.91]. In particular, we will look only at the degree $2i$ term in the above expression, $\frac{1}{i!}\mathrm{str}(D+L)^{2i}$, which are represented by invariant polynomials to define the Chern-Simons classes.

In the above situation if in addition $\nabla_1 = D+L$ is flat, then we want to define uniquely determined Chern-Simons classes of (E, ∇_t) which are independent of t.

For this purpose, we look at the *Character diagram* of Simons and Sullivan (see [27]):

Lemma 3.3. *Suppose (F, ∇) is a smooth connection on a manifold M. Then there is a uniquely determined differential character*

$$\widehat{c_k}(F, \nabla) \in \widehat{H^k}(M)_{\mathbb{R}}$$

which lifts the k-th Chern form defined in $A^k_{\mathbb{Z}}$. Furthermore, if ∇ is flat then $c_k(F) \in H^{2k}(M, \mathbb{Z})$ vanishes in $H^{2k}(M, \mathbb{R})$ and there is a unique lifting $CS_k(F, \nabla) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ of the integral class $c_k(F)$, for every $k > 0$.

Proof. The vanishing of the Chern form for a flat connection is by the Chern-Weil theory. The rest of the assertion is by the Chern-Simons-Cheeger construction [7] of differential characters. □

Consider the total Chern class

$$c(F) = 1 + c_1(F) + ... + c_f(F)$$

and the total Segre class

$$s(F) = 1 + s_1(F) + ... + s_f(F).$$

Then we have the relations

$$s(F) = \frac{1}{c(F)} \tag{5}$$

and

$$c(F - G) = c(F) \cdot s(G) \tag{6}$$

where G is any other vector bundle. These relations also hold if we replace the classes $c_i(F)$ by $\widehat{c_i}(F, \nabla)$ and $s_i(F)$ by $\widehat{s_i}(F, \nabla)$ which are defined by the relation (5). See [7, p.64-65].

Our goal is to define a canonical lifting in $\widehat{H^*}(M)_{\mathbb{R}}$ of the supertrace form (4) associated to a superconnection (E, ∇).

To motivate the definition we firstly look at the superconnection of the type $D+L$ where D preserves the grading and L is an odd endomorphism of the complex supervector bundle E. We consider the family of superconnection $\nabla_t = D + t \cdot L$ as above. We will see how the class is represented in the de Rham cohomology. Consider the product manifold $\mathbb{R} \times M$ and the pullback $pr_2^* E$ of the bundle E. This bundle is equipped with a superconnection

$$\bar{D} := dt\partial_t + D'$$

whose restriction to $\{t\} \times M$ is D_t. In terms of local trivialization of $E = M \times V$ we can describe $\bar{D}$, D' as follows: Write $\nabla_t = d_M + \theta_t$, where θ_t is a family of one-forms on M with values in End$V$; let θ be the form on $\mathbb{R} \times M$ not involving dt restricting to θ_t on $\{t\} \times M$. Then $D' = d_M + \theta$ and

$$\bar{D} = dt\partial_t + D' = (dt\partial_t + d_M) + \theta = d_{\mathbb{R}\times M} + \theta.$$

See also [21, p.91].

By the homotopy property of de Rham cohomology it follows that the class of $\mathrm{str}\nabla_t^{2n}$ in $H^{2n}(M, \mathbb{R})$ is independent of t.

Definition 3.4. *Suppose that there is a superconnection $D = D_0 \oplus D_1$ on the supervector bundle $E = E_0 \oplus E_1$ which preserves the $\mathbb{Z}/2\mathbb{Z}$-grading and the individual connections D_0 and D_1 are smooth connections on E_0 and E_1 respectively. Then we define the differential character of D to be equal to*

$$\widehat{c_k}(E, D) = \sum_{p+q=k} \widehat{c_p}(E_0, D_0) \cdot \widehat{s_q}(E_1, D_1) \tag{7}$$

Lemma 3.5. *The differential character $\widehat{c_k}(E, D)$ projects to the Chern form of D.*

Proof. By Quillen's theorem (see Proposition 3.2) the Chern character of E is the difference $ch(E_0) - ch(E_1)$. This means that the total Chern class $c(E)$ of E is the total Chern class of $E_0 - E_1$ in the de Rham cohomology. Since the differential characters of D_0 and D_1 project to their respective Chern forms in the exact sequence (3), the differential character of D which is given by the expression in (7) also projects to its Chern form.

□

To define the Chern-Simons class for any flat superconnection of the type $D + L$ we consider the Chern character of $D + L$:

$$\mathrm{ch(D + L) = str\ e^{(D+L)^2}}.$$

We look at the degree $2i$-terms of this expression

$$\mathrm{ch_i(D + L) := (str\ e^{(D+L)^2})_{2i}}$$

and we want to lift $\mathrm{ch_i(D + L)}$ as a differential character. Notice that there is a $GL(r, s)$-invariant polynomial, P_i, defined by $P_i(A_1, A_2, ..., A_i) := \mathrm{str}(A_1.A_2...A_i)$ for supermatrices $A_j \in M(r, s)$, such that the term $\mathrm{ch_i(D + L)}$ is obtained by the formula

$$\mathrm{ch_i(D + L) = \frac{1}{i!}.P_i((D + L)^2, ..., (D + L)^2)}. \tag{8}$$

Remark 3.6. *The Cheeger-Simons theory associates a differential character to an invariant polynomial. In the setting of vector bundles on manifolds, the algebra of invariant polynomials is generated by the polynomials $tr(A^i)$; whereas, as mentioned in* [5]*, in the supersetting the algebra of invariant functions is generated by $str(A^i)$. In particular, an invariant polynomial is expressed as a polynomial in terms of the standard generators. If* *for any other invariant polynomial G, the form obtained by substituting the curvature form $(D+L)^2$ in G is a closed form, then we can define the differential character associated to G simply by replacing P_i by G in the following discussion.*

Lemma 3.7. *Suppose $\{\nabla_t\}_t$ is a family of superconnections on a complex supervector bundle E such that ∇_0 is a $\mathbb{Z}/2\mathbb{Z}$-graded connection. Then we define the k-th differential character of (E, ∇_t) (equivalently, $\widehat{\mathrm{ch}_\mathrm{k}}(\nabla_t)$) in the ring of differential characters. Moreover, the differential character projects to the k-th Chern form of the superconnection ∇_t.*

Proof. Firstly, since ∇_0 preserves the $\mathbb{Z}/2\mathbb{Z}$-grading on E it corresponds to smooth connections D_0 and D_1 on the component bundles of $E = E_0 \oplus E_1$. Hence the differential character $\widehat{c_k}(E, \nabla_0)$ is defined by the expression (see (7))

$$\widehat{c_k}(E, \nabla_0) = \sum_{p+q=k} \widehat{c_p}(E_0, D_0) \cdot \widehat{s_q}(E_1, D_1).$$

Similarly, we can define the k-degree term of the Chern character in terms of the Chern characters of E_0 and E_1:

$$\widehat{\mathrm{ch}_k}(E, \nabla_0) := \widehat{\mathrm{ch}_k}(E_0, D_0) - \widehat{\mathrm{ch}_k}(E_1, D_1).$$

To define $\widehat{\mathrm{ch}_k}(E, \nabla_t)$, for $t \neq 0$, we can use the variational formula of differential characters of Cheeger-Simons [7, Proposition], obtained by looking at the polynomial P_k which defines $\widehat{\mathrm{ch}_k}(E, \nabla_1)$ (see (8)):

$$\widehat{\mathrm{ch}_k}(E, \nabla_1) := \frac{1}{(i-1)!}.\int_0^1 P_k(\frac{d}{dt}\nabla_t \wedge \nabla_t^{2(k-1)})dt]_{Z_{2k-1}} + \widehat{\mathrm{ch}_k}(E, \nabla_0). \tag{9}$$

This defines $\widehat{\mathrm{ch}_k}(E, \nabla_1)$ and similarly for any t. By well-known formulas we obtain an expression for $\widehat{c_k}(E, \nabla_t)$ also from $\widehat{\mathrm{ch}_k}(E, \nabla_t)$ in the ring of differential characters. □

Corollary 3.8. *With notations as above suppose $\{\nabla_t\}_t$ is a family of superconnections on E such that when ∇_0 preserves the grading. Assume that ∇_{t_0} is flat for some t_0. Then there is a uniquely determined Chern-Cheeger-Simons class*

$$CS_k(E, \nabla_{t_0}) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z}),$$

for $k \geq 2$.

Proof. Using Lemma 3.7, we can define $\widehat{c_k}(E, , \nabla_{t_0}) \in \widehat{H^{2k}}(M)_\mathbb{R}$. Since ∇_{t_0} is flat the Chern form is zero and the Character diagram in [27] or (3) defines a Chern-Cheeger-Simons class. □

Corollary 3.9. *Suppose $\nabla = D + L$ is flat superconnection on $E = E_0 \oplus E_1$ such that D preserves the grading. Then there is a uniquely determined class*

$$CS_k(E, D) = CS_k(E, D + L) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z}),$$

for $k \geq 2$.

Proof. Write a family of superconnections $\nabla_t := D + t.L$ on E, for $t \geq 0$. Now apply Corollary 3.8 directly to obtain the claim. □

We can extend the question of Cheeger and Simons as follows:

Question 3.10. *Suppose M is a smooth manifold and $(E, D + L)$ is a complex flat superconnection on M such that its Chern-Cheeger-Simons classes are defined. Are the classes*

$$CS_k(E, D + L) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$$

torsion if $k \geq 2$?

We will see some special situations in the next subsection where this question has a positive answer.

3.3. **Chern Simons classes for a morphism between flat connections.** Consider a homomorphism $u : E_0 \to E_1$ between complex vector bundles on a smooth manifold M. If u is an isomorphism over a subset $S \subset M$ then u determines a class in the relative K-group $K(M, S)$. A new way of constructing the Chern character in the relative de Rham cohomology of such a class is presented by Quillen [21]. To construct differential characters for u when E_0 and E_1 are equipped with a smooth connection, we will need to define a relative differential cohomology, which we do not look into in this paper. In the discussion below, we propose a definition of the Chern-Simons class for u when E_0 and E_1 are equipped with flat connections, in the absolute $\mathbb{R}/\mathbb{Z}$-cohomology of M.

Let

$$L = i \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}. \tag{10}$$

Here u^* is the adjoint of u relative to a given metric (see [21]). Regard $E = E_0 \oplus E_1$ as a complex supervector bundle on M and $D = D_0 + D_1$ be a superconnection on E preserving the $\mathbb{Z}/2\mathbb{Z}$-grading. Then L is an odd degree endomorphism of E and as shown in [21], $D + L$ is a superconnection and its curvature form $F = (D + L)^2$ is an even form with values in End$E$.

Definition 3.11. *Suppose (E_0, D_0) and (E_1, D_1) are flat connections and u and L are as above. Then we can define the Chern-Cheeger-Simons classes of the morphism u (which need not be a flat morphism) in the $\mathbb{R}/\mathbb{Z}$-cohomology of M by setting*

$$CS_k(u) := CS_k(E, D + L) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z}) \tag{11}$$

for $k \geq 1$.

We look at the following superconnection of the type $D + L$ considered by Quillen.

Theorem 3.12. (Application of Reznikov's theorem) *Suppose* $u : E_0 \to E_1$ *is a (not necessarily flat) homomorphism between flat complex vector bundles* (E_0, D_0) *and* (E_1, D_1) *on a compact Kähler manifold* M *or a smooth complex quasi–projective variety* M*. Then the classes*

$$CS_k(u) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Q})$$

are zero, for $k \geq 2$*.*

Proof. By Corollary 3.9 and definition 3.11 we have the equality

$$CS_k(u) = CS_k(E, D + L) = CS_k(E, D).$$

By definition 3.4, we have the explicit expression of the class

$$CS_k(u) = \sum_{p+q=n} CS_p(E_0, D_0).\widehat{s_q}(E_1, D_1).$$

When M is a compact Kähler manifold then Reznikov's theorem [24] says that

$$CS_i(E_0, D_0),\ CS_i(E_1, D_1) \in H^{2i-1}(M, \mathbb{R}/\mathbb{Z})$$

are torsion, if $i \geq 2$. A similar result is true if M is a smooth complex quasi–projective variety, by [12]. Since the classes $\widehat{s_q}$ are expressed in terms of CS_i for $i \leq q$ the assertion follows. This proves the theorem □

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