Line bundles of type $(1, ..., 1, 2, ..., 2, 4, ..., 4)$ on Abelian Varieties

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Abstract

We show birationality of the morphism associated to line bundles L of type $(1, ..., 1, 2, ..., 2, 4, ..., 4)$ on a generic $g-$dimensional abelian variety into its complete linear system such that $h^0(L) = 2^g$. When $g = 3$, we describe the image of the abelian threefold and from the geometry of the moduli space $SU_C(2)$ in the linear system $|2\theta_C|$, we obtain analogous results in $I\!PH^0(L)$.

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1 Introduction

Let L be an ample line bundle of type $\delta = (\delta_1, \delta_2, ..., \delta_g)$ on a g-dimensional abelian variety A. Consider the associated rational map $\phi_L : A \longrightarrow I\!PH^0(A, L)$.

When $g = 2$, Birkenhake, Lange and van Straten (see [3]) have studied line bundles of type $(1, 4)$ on abelian surfaces. Suppose L is an ample line bundle of type $(1, 4)$ on an abelian surface A. Then there is a cyclic covering $\pi : A \longrightarrow B$ of degree 4 and a line bundle M on B such that $\pi^*M = L$. Let X denote the unique divisor in $|M|$ and put $Y = \pi^{-1}(X)$. Their main theorem is

Theorem 1.1 *1) $\phi_L : A \longrightarrow A' \subset I\!P^3$ is birational onto a singular octic A' in $I\!P^3$ if and only if X and Y do not admit elliptic involutions compatible with the action of the Galois group of π.*

2)In the exceptional case $\phi_L : A \longrightarrow A' \subset I\!P^3$ is a double covering of a singular quartic A', which is birational to an elliptic scroll.

Here we generalise this situation to higher dimensions and show

Theorem 1.2 *Suppose L is an ample line bundle of type $\delta = (1, ..., 1, 2, ..., 2, 4, ..., 4)$ on a g-dimensional abelian variety A, $g \geq 3$, such that 1 and 4 occur equally often and atleast once in δ. Then, for a generic pair (A, L), the following holds.*

a) The associated morphism $\phi_L : A \longrightarrow I\!PH^0(A, L)$ is birational onto its image.

b) When $g = 3$, the image $\phi_L(A)$, can be described as follows,

there are 4 curves C_i on the image $\phi_L(A)$ such that the restricted morphism $\phi_L : \phi_L^{-1}(C_i) \longrightarrow C_i \subset \phi_L(A)$ is of degree 2.

Birkenhake et.al (see [3], Proposition 1.7, p.631) have shown the existence of the following commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\phi_L} & \phi_L(A) & \subset & I\!P^3 = I\!PH^0(L) \\ \downarrow\pi & & \downarrow & & \downarrow p \\ B & \xrightarrow{\phi_{M^2}} & \mathcal{K}(B) & \subset & I\!P^3 = I\!PH^0(M^2) \end{array}$$

where $p(z_0 : z_1 : z_1 : z_3) = (z_0^2 : z_1^2 : z_2^2 : z_3^2)$ and the pair (B, M) is a principally polarized abelian surface. This diagram explains the geometry of the image $\phi_L(A)$ from the geometry of the Kummer surface $\mathcal{K}(B)$ and it also gives the explicit equation of the surface $\phi_L(A)$ in $I\!P^3$.

Similarly, when $g \geq 3$ and the pair (A, L) as in 1.2, we show that there is a commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\phi_L} & \phi_L(A) & \subset & I\!P^{2^g-1} = I\!PH^0(L \\ \downarrow\pi & & \downarrow & & \downarrow p \\ B & \xrightarrow{\phi_{M^2}} & \mathcal{K}(B) & \subset & I\!P^{2^g-1} = I\!PH^0(M^2) \end{array}$$

where $p(z_0 : ... : z_{2^g-1}) = (z_0^2 : ... : z_{2^g-1}^2)$ and π is an isogeny of degree 2^g and the pair (B, M) is a principally polarized abelian variety. This will explain the birationality of the map ϕ_L and the geometry of the image $\phi_L(A)$, when $g = 3$, as asserted in 1.2. Since $deg(\phi_{M^2} \circ \pi) = 2^{g+1}$ and from the birationality of ϕ_L, it follows that $deg(p|_{\phi_L(A)}) = 2^{g+1}$. But since $degp = 2^{2^g-1}$ the inverse image of the Kummer variety in $I\!PH^0(L)$ has components other than the image $\phi_L(A)$. Hence the image $\phi_L(A)$ will be defined by forms other than those coming from those forms which define the variety $\mathcal{K}(B)$.

We study the situation when $g = 3$, in detail. Consider a pair (A, L), with L being an ample line bundle of type $(1,2,4)$ on an abelian threefold A. Consider an isogeny $A \longrightarrow B = A/G$, where G is a maximal isotropic subgroup of $K(L)$ of the type $\frac{\mathbf{Z}}{2\mathbf{Z}} \times \frac{\mathbf{Z}}{2\mathbf{Z}} \times \frac{\mathbf{Z}}{2\mathbf{Z}}$. Then B is a principally polarized abelian threefold. If B is isomorphic to the Jacobian variety of C, $J(C)$, where C is a smooth non-hyperelliptic curve of genus 3, then the situation becomes interesting because of the following results due to Narasimhan and Ramanan.

Theorem 1.3 *(See [12], Main Theorem, p.416) If C is a non-hyperelliptic curve of genus 3, then the moduli space $SU_C(2)$ is isomorphic to a quartic hypersurface in $I\!P^7$.*

(Here $I\!P^7 = |2\theta|$, where θ is the canonical principal polarization on the Jacobian $J(C)$ and $SU_C(2)$ is the moduli space of rank 2 semi-stable vector bundles with trivial determinant on the curve C).

Theorem 1.4 *(See [11]) The Kummer variety $\mathcal{K}$ is precisely the singular locus of $SU_C(2)$, if $g(C) \geq 3$.*

The quartic hypersurface, $F = 0$, is classically called the *Coble quartic* and is $\mathcal{G}(2\theta)$-invariant in the linear system $|2\theta|$. We identify the group of projective transformations, H, of order 8, which acts on $\pi^{-1}\mathcal{K}(C)$, (see 3.7). The $\mathcal{G}(L)$-invariant octic hypersurface R, given as $F(z_0^2 : ... : z_7^2) = 0$ in $I\!PH^0(L)$, then contains the components $h(\phi_L(A)), h \in H$ in its singular locus.

Now we use the geometry of the moduli space $SU_C(2)$ in the linear system $|2\theta|$, which has been extensively studied (see [5], for instance), to get analogous results in $I\!PH^0(L)$.

We show

Theorem 1.5 *Consider a pair (A, L), as above. Let $a \in K(L)$ be an element of order 2 such that $e^L(a, g) = -1$, for all $g \in G$, (here e^L is the Weil form on the group $K(L)$). Let $I\!PW_a$ be an eigenspace in $I\!PH^0(L)$, for the action of a. Then there is a polarized abelian surface (Z, N), N is ample of type $(1,4)$ and a commutative diagram*

$$\begin{array}{ccccccc} Z & \xrightarrow{\phi_N} & \phi_N(Z) & \subset & I\!PH^0(N) \simeq & I\!PW_a \\ \downarrow f & & \downarrow & & \downarrow q & \downarrow p \\ P_a & \xrightarrow{\phi_{2\theta_a}} & \mathcal{K}(P_a) & \subset & I\!PH^0(2\theta_a) \simeq & I\!PV_a \end{array}$$

Here (P_a, θ_a) is the Prym variety associated to the 2-sheeted unramified cover of the curve C, given by $\pi(a)$ and $\mathbb{P}V_a$ is the eigenspace in $\mathbb{P}H^0(2\theta)$, for the action of $\pi(a)$. The isomorphisms above are Heisenberg equivariant and the morphism q is given as $(r_0 : r_1 : r_2 : r_3) \mapsto (r_0^2 : r_1^2 : r_2^2 : r_3^2)$.

We thus obtain the situation described by Birkenhake et.al in the case $g = 2$, nested in the case $g = 3$.

Moreover, the $\mathcal{G}(N)$-invariant octic surface $\phi_N(Z)$ is mapped isomorphically onto the $a^{\perp}/a(\simeq Heis(4))$-octic $R \cap \mathbb{P}W_a$ and we identify the set $\cap_{h \in H} h(\phi_L(A))$ with the set of all pinch points and the coordinate points in $\phi_N(Z)$, occurring in each of the eigenspace $\mathbb{P}W_a$, (see 5.6). Finally, we make some remarks on the moduli space $\mathcal{A}^{(1,2,4)}$.

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Notation : Suppose L is a symmetric line bundle i.e. $L \simeq i^*L$ for the involution $i : A \longrightarrow A$, $a \mapsto -a$.

The *fixed group* of L is $K(L) = \{a \in A : L \simeq t_a^*L\}$, $t_a : A \longrightarrow A, x \mapsto a + x$.

The *theta group* of L is $\mathcal{G}(L) = \{(a, \phi) : L \overset{\phi}{\simeq} t_a^*L\}$.

$K_1(\delta) = \frac{\mathbb{Z}}{d_1\mathbb{Z}} \times ... \times \frac{\mathbb{Z}}{d_g\mathbb{Z}}$, and $\widehat{K_1(\delta)} = Hom(K_1(\delta), \mathbb{C}^*)$.

The *Heisenberg group* of type δ, $Heis(\delta) = \mathbb{C}^* \times K_1(\delta) \times \widehat{K_1(\delta)}$ and $V(\delta) = \{f : f : K_1(\delta) \longrightarrow \mathbb{C}\}$.

The *Weil form* $e^L : K(L) \times K(L) \longrightarrow \mathbb{C}^*$, is the commutator map $(x, y) \mapsto x'y'x'^{-1}y'^{-1}$, for any lifts $x', y' \in \mathcal{G}(L)$ of $x, y \in K(L)$.

For any $a \in K(L)$, $a^{\perp} = \{x \in K(L) : e^L(a, x) = 1\}$.

Consider the semi-direct product, $\mathcal{G}(L) \ltimes (i)$, of the theta group associated to L and the group generated by the involution i. Let $\gamma \in \mathcal{G}(L) \ltimes (i)$ be an element of order 2.

$H^0(L)^{\pm}_{\gamma} = (\pm 1)-$eigenspace of $H^0(L)$ for the action of γ.

$h^0(L)^{\pm}_{\gamma} = dim H^0(L)^{\pm}_{\gamma}$.

$Q(V)$ = function field of a variety V.

2 Birationality of the map ϕ_L.

Let L be an ample line bundle of type $\delta = (1, ..2, .., 4)$ on a g-dimensional abelian variety A. Here number of 2's= number of 4's in δ. Let $K(L) = \{a \in A : t_a^* L \simeq L\}$, where t_a denotes translation by a on A. Choose a maximal isotropic subgroup G of $K(L)$ w.r.t. the Weil form e^L, containing $2K(L)$ and having only elements of order 2. Then $G \simeq \frac{\mathbb{Z}}{2\mathbb{Z}} \times ... \times \frac{\mathbb{Z}}{2\mathbb{Z}}$, g-times. Consider the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L) \longrightarrow K(L) \longrightarrow 0.$$

Let G' be a lift of G in $\mathcal{G}(L)$. Consider the isogeny $A \xrightarrow{\pi} B = A/G$. Then L descends to a principal polarization M on B. By Projection formula and using the fact that $\pi_* \mathcal{O}_A = \oplus_{\chi \in \hat{G}} L_\chi$, where L_χ denotes the line bundle corresponding to the character χ, we deduce that

$$H^0(L) = \oplus_{\chi \in \hat{G}} H^0(M \otimes L_\chi).$$

Hence $\{s_\chi \in H^0(M \otimes L_\chi) : \chi \in \hat{G}\}$ is a basis for the vector space $H^0(L)$ and since $M^2 \otimes L_\chi^2 \simeq M^2$, $s_\chi^2 = s_\chi \otimes s_\chi \in H^0(M^2) \forall \chi \in \hat{G}$.

Consider the homomorphism $\epsilon_2 : \mathcal{G}(L) \longrightarrow \mathcal{G}(L^2), (x, \phi) \mapsto (x, \phi^{\otimes 2})$ and the inclusion $K(L) \subset K(L^2)$.

Then the subgroup $G \subset K(L^2)$ is isotropic for the Weil form e^{L^2}. Moreover, if $x \in K(L)$ and $g \in G$, then

$$e^{L^2}(x, g) = e^L(x, g).e^L(x, g) = 1.$$

Hence $\epsilon_2(\mathcal{G}(L)) \subset \mathcal{Z}(\epsilon_2(G'))$ and $\pi(K(L)) \subset K(M^2)$. (Here $\mathcal{Z}(\epsilon_2(G')) = \{a \in \mathcal{G}(L^2) : a.g' = g'.a, \forall g' \in \epsilon_2(G')\}$).

Now $\mathcal{G}(M^2) = \mathcal{Z}(\epsilon_2(G'))/\epsilon_2(G')$ and $H^0(M^2) = H^0(L^2)^{G'}$, where $H^0(L^2)^{G'}$ denotes the vector subspace of $\epsilon_2(G')$-fixed sections of $H^0(L^2)$. For $g' \in G'$ and $\chi \in \hat{G}$, $g'(s_\chi^2) = \chi^2(g).s_\chi^2 = s_\chi^2$. Hence $s_\chi^2 \in H^0(L^2)^{G'}$, for all $\chi \in \hat{G}$.

We now show that $\{s_\chi^2 : \chi \in \widehat{G}\}$ is a basis for $H^0(M^2)$, for a generic pair (A, L)..

In fact, we show that the homomorphism

$$\sum_{\chi \in \hat{G}} H^0(M \otimes L_\chi).H^0(M \otimes L_\chi) \xrightarrow{\rho} H^0(M^2)...(*)$$

is an isomorphism, for a generic pair (A, L).

Consider the pair $(A,L) = (E_1 \times ... \times E_r \times, A_1 \times ...A_s, p_1^* L_1 \otimes ... \otimes p_{r+s}^* L_{r+s})$, where r is the number of 2's occurring in δ, $E_1,...,E_r$ are elliptic curves with line bundles L_i on E_i of degree 2 and A_j are simple abelian surfaces with line bundles L_j on A_j of type $(1,4)$ (by 1.1, $\phi_{L_j}(A_j) \subset |L_j|$ is an octic surface).

In this case, one can easily see that the homomorphism

$$S = Sym^2 H^0(L_1) \otimes ... \otimes Sym^2 H^0(L_{r+s}) \longrightarrow H^0(L_1^2) \otimes ... \otimes H^0(L_{r+s}^2) = H^0(L_1^2 \otimes ... \otimes L_{r+s}^2)$$

is injective. Here, $(B,M) = (F_1, M_1) \times ... \times (F_r, M_r) \times (B_1, M_1') \times ...(B_s, M_s')$, where (F_j, M_j) are polarised elliptic curves of degree 1 and (B_j, M_j) are principally polarised abelian surfaces. Also, the group G is generated by elements of the type $(e_1,..,e_r,a_{r+1}',..,a_g')$, where each of e_j and a_j' are non-trivial 2 torsion elements of E_j and A_j, respectively. Now it is easy to see that $\sum_{\chi \in \hat{G}} H^0(M \otimes L_\chi).H^0(M \otimes L_\chi) \subset S$ and $H^0(M^2) \subset H^0(L^2)$ and (*) is an isomorphism.

Hence, for a generic pair (A,L) as above, (*) is an isomorphism.

As a consequence, we obtain the following

Proposition 2.1 *Consider a generic principally polarized abelian variety (B', M') of dimension g. Let H be a subgroup of 2-torsion points of B', of order g. Then the image of H in $\mathcal{K}(B')$ generates the linear system $|2M'|$.*

(This is well known if H consists of all the 2-torsion points of B', for any principally polarised pair (B', M').)

Proof: Since the map $B' \xrightarrow{\phi_{2M'}} |2M'|$ is given by $a \mapsto t_a^*\theta + t_{-a}^*\theta$, where θ is the unique divisor in $|M'|$, the assertion is equivalent to showing the surjectivity of the multiplication map

$$\sum_{\chi \in \hat{H}} H^0(M' \otimes L_\chi) \otimes H^0(M' \otimes L_\chi) \xrightarrow{\rho} H^0(M'^2)..(!).$$

Here $\hat{H}$ is the dual image of H in $Pic^0(B')$. But we showed above this isomorphism, if $\hat{H}$ gives rise to a g-sheeted cover (A', L') of (B', M'), where L' is of type $(1,..,2,..,4)$. Otherwise, $\hat{H}$ gives a cover (A', L') where L' is of type $(2,2,...,2)$. By similar argument used in proving (*), (!) is still true when $A' = E_1 \times ... \times E_g$ and $L' = L_1 \times L_2 ... \times L_g$, where L_j are line bundles of degree 2 on the elliptic curves E_j. Hence our assertion is true for a generic pair (B', M'). □

So, for a generic pair (A,L), the map $I\!PH^0(L) \longrightarrow I\!PH^0(M^2)$, given as $(...,s_\chi,...) \mapsto (...,s_\chi^2,...)$ is a morphism and we obtain a commutative diagram (I),

$$\begin{array}{ccccc} A & \xrightarrow{\phi_L} & \phi_L(A) & \subset & I\!P^{2^g-1} = I\!PH^0(L) \\ \downarrow\pi & & \downarrow & & \downarrow p \\ B = A/G & \xrightarrow{\phi_{M^2}} & \mathcal{K}(B) & \subset & I\!P^{2^g-1} = I\!PH^0(M^2) \end{array}$$

where $p(...,s_\chi,...) = (...,s_\chi^2,...)$.

Remark 2.2 *Since $\phi_{M^2}\circ\pi$ is a morphism, ϕ_L is a morphism i.e. L is base point free.*

Lemma 2.3 *Consider a pair (A,L) as in 1.2. Let $\gamma \in \mathcal{G}(L)\ltimes(i)$ be an element of order 2. Then $H^0(L) \neq H^0(L)^\pm_\gamma$.*

Proof: Case 1: Suppose $\gamma = g \in \mathcal{G}(L)$. Then the action of γ is fixed point free on A. Hence by Atiyah- Bott fixed point theorem,

$$h^0(L)^+_\gamma = h^0(L)^-_\gamma = h^0(L)/2.$$

Case 2: Suppose $\gamma = i$. Then

$$h^0(L)^\pm_i = h^0(L)/2 \pm 2^{g-s-1}$$

(see [1], 4.6.6), where s is the number of odd integers occurring in the type of L.

Case 3: Suppose $\gamma = i.g$ and $H^0(L) = H^0(L)^+_\gamma$, where $g \in \mathcal{G}(L)$ is an element of order 2. Let $s \in H^0(L)^-_g$. Then $\gamma(s) = s$ gives $i(s) = -s$, i.e. $s \in H^0(L)^-_i$. Hence $H^0(L)^-_g \subset H^0(L)^-_i$. But this contradicts the fact that $h^0(L)^-_g = 2^{g-1}$ and $h^0(L)^-_i = 2^{g-1} - 2^{g-s-1}$ (here $s > 1$). Similarly $H^0(L) \neq H^0(L)^-_\gamma$. □

Suppose ϕ_L is not birational and is a finite morphism of degree d, $d > 1$. Notice that $A \overset{\phi_{M^2}\circ\pi}{\longrightarrow} \mathcal{K}(B)$ is a Galois covering with Galois group $(G,i) \simeq (\frac{\boldsymbol{Z}}{2\boldsymbol{Z}})^{g+1}$ and we have the extension of fields, $Q(\mathcal{K}(B)) \longrightarrow Q(\phi_L(A)) \longrightarrow Q(A)$. Hence the Galois group of $Q(A)$ over $Q(\phi_L(A))$ is a subgroup of (G,i), say H, of order d. Let $\gamma \in H$. Then γ is an involution on A, given as $a \mapsto \epsilon a + g$ where $\epsilon = \pm 1$, $g \in G$ and it induces an involution γ' on $H^0(L)$.

Hence ϕ_L factorizes as $A \xrightarrow{\psi_1} A/(\gamma) \xrightarrow{\psi_2} \phi_L(A) \subset I\!P^{2^g-1}$. This means that the morphism ψ_2 is given by the pair $(N, H^0(L)^+_{\gamma'})$ or $(N', H^0(L)^-_{\gamma'})$, where N and N' are line bundles on $A/(\gamma)$ whose pullback to A is L. By 2.3, $H^0(L) \neq H^0(L)^\pm_{\gamma'}$ and hence $\phi_L(A)$ is a degenerate variety in $I\!P^{2^g-1}$. This contradicts the fact that the morphism ϕ_L is given by a complete linear system. Hence ϕ_L is a birational morphism.

3 Configuration when $g = 3$

Assume $g = 3$. Choose a *theta structure* $f : \mathcal{G}(L) \longrightarrow Heis(2,4)$, (i.e. f is an isomorphism which restricts to identity on $\mathbb{C}^*$.) This induces an isomorphism $H^0(L) \simeq V(2,4)$ and a level structure $K(L) \simeq \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}}$. Let $\sigma_1, \tau_1, \sigma_2, \tau_2$ be the generators of the summands such that $o(\sigma_i) = 2$ and $o(\tau_i) = 4$. The Weil form e^L is given as

$$e^L(\sigma_1, \sigma_2) = -1$$

$$e^L(\tau_1, \tau_2) = -i$$

$$e^L(\sigma_i, \tau_j) = 1.$$

Then we see that the subgroup $G =< \sigma_1, \tau_1^2, \tau_2^2 >$ of $K(L)$ is maximal isotropic for the form e^L.

We may assume L is strongly symmetric (see [10], Remark 2.4., p.160), i.e., $e_*^L(g) = 1$ for all $g \in K(L)_2$, after choosing a normalized isomorphism $\psi : L \simeq i^*(L)$, i.e. $\psi(0) = +1$. Here $e_*^L : A_2 \longrightarrow \{\pm 1\}$ is a quadratic form whose value at an element a, of order 2 is the action of ψ at the fibre of L at a.

Consider the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L) \longrightarrow K(L) \longrightarrow 0$$

and the homomorphism $\delta_{-1} : \mathcal{G}(L) \longrightarrow \mathcal{G}(L)$, $z \mapsto izi$. Then $\delta_{-1}(z) = \alpha z^{-1}$ for some $\alpha \in \mathbb{C}^*$.

By [6], Proposition 2.3, p.141, we further assume that f is a *symmetric theta structure*, i.e. $f \circ \delta_{-1} = D_{-1} \circ f$, where $D_{-1} : Heis(\delta) \longrightarrow Heis(\delta)$ is the homomorphism $(\alpha, x, l) \mapsto (\alpha, -x, -l)$.

Lemma 3.1 *If $z \in \mathcal{G}(L)$ is an element of order 2 and $z \neq \pm 1$ then $\delta_{-1}(z) = e_*^L(z)z$.*

Proof: : See [8], Proposition 3, p.309. □

Remark 3.2 *Let $\sigma_1', \sigma_2', \tau_1', \tau_2' \in \mathcal{G}(L)$ be lifts of $\sigma_1, \sigma_2, \tau_1, \tau_2$ such that $o(\sigma_i') = 2, o(\tau_i') = 4$. Since $\tau_i^2 \in G$, $e_*^L(\tau_i^2) = 1$, hence by 3.1, $\delta_{-1}((\tau_i')^2) = (\tau_i')^2$. Hence $\delta_{-1}(\tau_i') = c.{\tau'}_i^{-1}, c = \pm 1$. We may assume $c = +1$, by suitably altering the lift τ_i'.*

Let $G' =< \sigma_1', (\tau_1')^2, (\tau_2')^2 >\subset \mathcal{G}(L)$.

Then L descends to a principal polarization M on $B = A/G$.

As remarked in Section 2,

$$H^0(L) = \oplus_{\chi\in\hat{G}} H^0(M \otimes L_\chi)$$

and $\{s_\chi \in H^0(M \otimes L_\chi), \chi \in \hat{G}\}$ form a basis of $H^0(L)$.

Consider the commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\psi_L} & Pic^0(A) \\ \downarrow\pi & & \uparrow\hat{\pi} \\ B & \xrightarrow{\psi_M} & Pic^0(B) \end{array}$$

where $\psi_L(a) = t_a^* L \otimes L^{-1}$ and $\psi_M(b) = t_b^* M \otimes M^{-1}$. Then ψ_M is an isomorphism and since $\hat{\pi}(L_\chi) = 0$, we have $\pi^{-1}\psi_M^{-1}(L_\chi) \in K(L) \forall \chi \in \hat{G}$. Hence $M \otimes L_\chi \simeq t_b^* M$ where $b \in \pi(K(L))$. The basis elements $\{s_\chi\}_{\chi\in\hat{G}}$ can be written as $s_0, s_1 = \sigma_2'(s_0), s_2 = \tau_1'(s_0), s_3 = \tau_2'(s_0), s_4 = \sigma_2'\tau_1'(s_0), s_5 = \sigma_2'\tau_2'(s_0), s_6 = \tau_1'\tau_2'(s_0), s_7 = \sigma_2'\tau_1'\tau_2'(s_0)$.

Lemma 3.3 *If $a \in K(L)_2$, then $a.i = i.a$.*

Proof: By 3.1, $\delta_{-1}(a) = e_*^L(a)a$. Since $e_*^L(a) = 1, a.i = i.a$. □

In particular, $g'i(s_0) = ig'(s_0)$, for all $g' \in G'$. Since $g's_0 = s_0$, $i(s_0) \in H^0(M)$. This implies that $i(s_0) = \pm s_0$. We may assume $i(s_0) = s_0$.

Lemma 3.4 *a) $i\sigma_2'(s_0) = \sigma_2'(s_0)$.*

b) $i\tau'_j(s_0) = \tau_j'(s_0)$.

c) $i\sigma_2'\tau_j'(s_0) = \sigma_2'\tau_j'(s_0)$.

d) $i\tau_1'\tau_2'(s_0) = -\tau_1'\tau_2'(s_0)$.

e)$i\sigma_2'\tau_1'\tau_2'(s_0) = -\sigma_2'\tau_1'\tau_2'(s_0)$

Proof: We will use 3.3 and the fact that $g'(s_0) = s_0$, for all $g' \in G'$.

a) $i\sigma_2'(s_0) = \sigma_2' i(s_0) = \sigma_2'(s_0)$.

b) $i\tau'_j(s_0) = \tau_j'^{-1} i(s_0) = \tau_j'^3(s_0) = \tau_j'(s_0)$, (since $\tau_j'^2 \in G'$).

c) $i\sigma_2'\tau_j'(s_0) = \sigma_2' i\tau_j'(s_0) = \sigma_2'\tau_j'(s_0)$.

d) $i\tau_1'\tau_2'(s_0) = \tau_1'^{-1} i\tau_2'(s_0) = \tau_1'\tau_1'^2\tau_2'(s_0) = -\tau_1'\tau_2'\tau_1'^2(s_0) = -\tau_1'\tau_2'(s_0)$ (since $e^L(\tau_1'^2, \tau_2') = -1, \tau_1'^2 \in G'$).

e) $i\sigma_2'\tau_1'\tau_2'(s_0) = \sigma_2' i\tau_1'\tau_2'(s_0) - \sigma_2'\tau_1'\tau_2'(s_0)$ □

Hence we have shown the following.

Proposition 3.5 *The vector subspace $H^0(L)_i^+$ of $H^0(L)$ is generated by the sections $s_0, s_1, s_2, s_3, s_4, s_5$ and the subspace $H^0(L)_i^-$ of $H^0(L)$ is generated by the sections s_6 and s_7.*

We then have the commutative diagram,

$$\begin{array}{ccccc} A & \xrightarrow{\phi_L} & \phi_L(A) & \subset & I\!P(H^0(L)) \\ \downarrow\pi & & \downarrow & & \downarrow p \\ B = A/G & \xrightarrow{\phi_{M^2}} & \mathcal{K}(B) & \subset & I\!P(H^0(M^2)) \end{array} \quad ...(\mathrm{I}).$$

Here $degree(p) = 2^7$ and $degree(\pi) = 8$. Since we have shown that ϕ_L is a birational morphism, $degree(\phi_L) = 1$ and hence $degree(p|_{\phi_L(A)}) = 2^4$. The ramification locus of $p|_{\phi_L(A)}$ is $\bigcup_{i=0}^{7}(H_i \cap \phi_L(A))$, where H_i is the hyperplane $\{s_i = 0\}$ in $I\!P(H^0(L))$, $0 \le i \le 7$.

Consider the group J generated by the projective transformations α_i,

$$(s_0, ..., s_i, ..., s_7) \mapsto (s_0, ..., -s_i, ..., s_7)$$

for $i = 1, ..., 7$.

Then $order(J) = 2^7$ and the group J is the Galois group of the finite morphism p.

Proposition 3.6 *The group $G' \times < i >$ can be identified as a subgroup of J.*

Proof: : Since the action of $g \in G$ on the abelian threefold is fixed point free, the ± 1-eigenspaces of $H^0(L)$ under the transformation $g \in G'$ are equidimensional. Also, $g(s_\chi) = \chi(g).s_\chi$, for all $\chi \in \hat{G}$, implies that $g = \alpha_i\alpha_j\alpha_k\alpha_l \in J$, for some $0 \le i < j < k < l \le 7$. Here $\alpha_0 = \alpha_1\alpha_2...\alpha_7$. By 3.5, $i(s_0 : ... : s_7) = (s_0 : ... : s_5 : -s_6 : -s_7)$. Hence the involution $i = \alpha_6.\alpha_7$. Hence we can identify $G' \times < i >$ as a subgroup of J. □

Moreover, since the Galois group of the morphism p, $Gal(p) = J$ and the subgroup $G' \times < i > \subset J$, leaves the image $\phi_L(A)$ invariant in $I\!PH^0(L)$, we have the following

Proposition 3.7 *Consider the commutative diagram (I). The inverse image of the variety, $\mathcal{K}(B)$, has eight distinct components $h(\phi_L(A))$, where $h \in J/(G' \times < i >)$.*

In Section 2, we have seen that $\{t_0 = s_0^2, t_1 = \sigma_2'(s_0^2), t_2 = \tau_1'(s_0^2), t_3 = \tau_2'(s_0^2), t_4 = \sigma_2'\tau_1'(s_0^2), t_5 = \sigma_2'\tau_2'(s_0^2), t_6 = \tau_1'\tau_2'(s_0^2), t_7 = \sigma_2'\tau_1'\tau_2'(s_0^2)\}$
form a basis of $H^0(M^2)$.

Remark 3.8 *(We use the same notations for the elements in $K(L)$ and their images in $K(M^2)$.) The elements $\sigma_2', \tau_1', \tau_2'$ of $\mathcal{G}(M^2)$ act on these sections as follows.*

	σ_2'	τ_1'	τ_2'
t_0	t_1	t_2	t_3
t_1	t_0	t_4	t_5
t_2	t_4	t_0	$-t_6$
t_3	t_5	t_6	t_0
t_4	t_2	t_1	$-t_7$
t_5	t_3	t_7	t_1
t_6	t_7	t_3	$-t_2$
t_7	t_6	t_5	$-t_4$

Now let $H_i = \{s_i = 0\}$ denote the coordinate hyperplanes in $I\!PH^0(L)$, for $i = 0, 1, ..., 7$. Consider the curve $C = H_6 \cap H_7 \cap \phi_L(A)$. Then the involution i acts trivially on the curve C and hence the degree of the restricted morphism $\phi_L^{-1}(C) \longrightarrow C$ is at least 2.

Proposition 3.9 *The restricted morphism* $\phi_L' : \phi_L^{-1}(C) \longrightarrow C$ *is of degree* 2.

Proof: : Consider the commutative diagram

$$\begin{array}{ccc} \phi_L^{-1}(C) & \xrightarrow{\phi_L'} & C \\ \downarrow \pi' & & \downarrow p' \\ \phi_{M^2}^{-1}(p(C)) & \xrightarrow{\phi_{M^2}'} & p(C) \end{array}$$

Suppose the degree of the restricted morphism ϕ_L' is greater than 2. Since the Galois group of the morphism $\phi_{M^2}' \circ \pi'$ is the group $G \times < i >$, the Galois group of ϕ_L' contains an element $g \in G$. Hence the element g acts trivially on the curve C. This means that C is contained in one of the eigenspaces $I\!PW^{\pm}$ of $I\!PH^0(L)$, for the action of g. We claim that the intersection $\phi_L(A) \cap I\!PW^{\pm}$ is at most a finite set of points. This will give a contradiction.

If $g^{\perp} = \{a \in K(L) : e^L(a,g) = 1\}$, then $\frac{g^{\perp}}{<g>} \simeq Heis(1,1,4)$ or $Heis(1,2,2)$ and the group $\frac{g^{\perp}}{<g>}$ acts on the linear space $I\!PW^{\pm}$. Hence projecting from $I\!PW^{\pm}$ gives a map $\phi_g : \frac{A}{<g>} \longrightarrow I\!PW^{\mp}$, which is base point free in the first case (by [2]) and has a finite base locus in the second case (by [10]). This proves our claim. □

Now, the group G leaves the curve C invariant and moreover since $\sigma_2(H_6) = H_7$, we get $\sigma_2(C) = C$. Hence the curves

$$\tau_1(C) = H_3 \cap H_5 \cap \phi_L(A)$$

$$\tau_2(C) = H_2 \cap H_4 \cap \phi_L(A)$$

$$\tau_1.\tau_2(C) = H_0 \cap H_1 \cap \phi_L(A)$$

are also invariant for the action of σ_2 and since for $x \in C$, $i(x) = x$, $i.\tau_j^2(\tau_j(x)) = \tau_j^2.\tau_j^{-1}i(x) = \tau_j(x)$. By $K(L)$-invariance of the image $\phi_L(A)$, we get

Corollary 3.10 *The morphism ϕ_L restricts to a morphism of degree 2 on the curves $\phi_L^{-1}(C)$, $\phi_L^{-1}(\tau_1(C))$, $\phi_L^{-1}(\tau_2(C))$ and $\phi_L^{-1}(\tau_1.\tau_2(C))$, onto their respective images. Moreover, the Galois groups of these restricted morphisms are $< i >$, $< i.\tau_1^2 >$, $< i.\tau_2^2 >$ and $< i.\tau_1^2.\tau_2^2 >$, respectively.*

Let A_2^+ denote the set of points of order 2 on A where the involution i acts on the fibre of L at those points as $+1$ and A_2^- denote the set of points where i acts as -1. By [1], Remark 4.7.7, $cardinality(A_2^+) = 48$ and $cardinality(A_2^-) = 16$. Hence if $a \in A_2^-$ and $s \in H^0(L)_i^+$, then $s(a) = 0$. This implies that for $a \in A_2^-, \phi_L(a) = (0 : 0 : ... : 0 : c_1 : c_2) \in \mathbb{P}H^0(L)$, for some $c_1, c_2 \in \mathbb{C}$.

Proposition 3.11 *Let $a \in A_2^+$(respectively A_2^-) and $g \in K(L)_2$. Then $a + g \in A_2^+$(respectively A_2^-).*

Proof: : Let $g \in K(L)_2$ and $(g, \phi) \in \mathcal{G}(L)$ be a lift of order 2 and $\psi : L \longrightarrow i^*(L)$ be the normalized isomorphism. By [7], Proposition 3, p.309,

$$\delta_{-1}(g, \phi) = (g, (t_g^*\psi)^{-1} \circ i^*\phi \circ \psi)$$

$$= e_*^L(g).(g, \phi)$$

$$= (g, \phi)\,(since\,L\,is\,strongly\,symmetric).$$

Hence the following diagram commutes

$$\begin{array}{ccc} L & \overset{\psi}{\simeq} & i^*(L) \\ \downarrow\phi & & \downarrow i^*(\phi) \\ t_g^*L & \overset{t_g^*(\psi)}{\simeq} & i^*t_g^*L = t_g^*(i^*L) \end{array}$$

Evaluating at $a \in A_2^+$(respectively A_2^-), gives $\psi(a) = t_g^*(\psi)(a) = \psi(a + g)$, i.e. $a + g \in A_2^+$(respectively A_2^-). □

Now let $a \in A_2^-$ then $\phi_L(a) = (0 : ... : c_1, c_2)$ for some $c_1, c_2 \in \mathbb{C}$. Then $\sigma_2\phi_L(a) = (0 : ... : c_2 : c_1)$. We may assume $c_2 \neq 0$. Let $P_0 = \phi_L(a) = (0 : ... : c : 1)$ and $Q_0 = p(P_0) = (0 : ... : c^2 : 1)$, for some $c \in \mathbb{C}$.

Proposition 3.12 *The points $h(P_0)$, $h \in K(L)/ < \tau_1^2, \tau_2^2 >$ are of degree 4 on the image $\phi_L(A)$.*

Proof: : By 3.11, the action of G on the set A_2^- has two distinct orbits, namely $O_1 = \{a + g : g \in G\}$ and $O_2 = \{a + \sigma_2 + g : g \in G\}$. Then $\phi_{M^2} \circ \pi(O_1) = Q_0$ and $\phi_{M^2} \circ \pi(O_2) = \sigma_2(Q_0)$. Notice that $P_0 \in \tau_1(C) \cap \tau_2(C) \cap \tau_1.\tau_2(C)$. Hence, by 3.10, $\phi_L^{-1}(P_0) = \{a, a + 2\tau_1, a + 2\tau_2, a + 2\tau_1 + 2\tau_2\}$. The assertion now follows from the $K(L)$-invariance of the image $\phi_L(A)$. □

Corollary 3.13 *The points $b(Q_0)$, where $b \in < \pi(\sigma_2), \pi(\tau_1), \pi(\tau_2) >$, lie on the Kummer $\mathcal{K}(B)$.*

4 Prym Varieties

We recall few facts on Prym varieties (see [5], [9], [12], for details).

Let C be a smooth projective curve of genus g. We will assume C has no vanishing theta nulls. In particular, when $g = 3$, this means C is a non-hyperelliptic curve. A point of order 2, in $X = Jac(C)$, say x, defines an unramified 2- sheeted cover C_x of C, $q_x : C_x \longrightarrow C$. Let $P_x = Ker(Nm(q_x) : Jac(C_x) \longrightarrow X)^o$, where '$o$' denotes the connected component containing $0 \in Jac(C_x)$. Here $Nm(q_x)(\mathcal{O}(\sum r_i P_i)) = \mathcal{O}(\sum r_i q_x(P_i))$ is the norm map. This defines a principally polarized abelian variety (P_x, θ_{P_x}), of dimension $g - 1$. Since the kernel of the dual map $q_x^{'} : X \longrightarrow Jac(C_x)$ is generated by the element x, $q_x{}'$ induces an isomorphism $x^{\perp}/x \longrightarrow P_x[2]$. Since $q_{x*}\mathcal{O}_{C_x} \simeq \mathcal{O}_C \oplus x$, we have $det q_{x*}\mathcal{O}_{C_x} \simeq x$. Hence $det(q_{x*}(p))$ is also x, for any $p \in ker(Nm(q_x))$.

Fix a $z \in X$ with $z^2 \simeq x$. This gives a map

$$\psi_x : Ker(Nm(q_x)) \simeq P_x \cup P_x \longrightarrow SU_C(2).$$

where $\psi_x(p) = (q_{x*}p) \otimes z$.

The image of ψ_x is independent of the choice of z. Recall the map

$$SU_C(2) \xrightarrow{\phi} |2\theta_C| \simeq \mathbb{P}(H^0(SU_C(2), \mathcal{L}))$$

where $\mathcal{L}$ generates $Pic(SU_C(2)) \simeq \mathbb{Z}$.

Let $\mathbb{P}V_x^+$ and $\mathbb{P}V_x^-$ be the two eigenspaces for the action of x on $|2\theta_C|$. Then there is one component of $Ker(Nm(q_x))$ in each eigenspace. So we get a map $\phi_x : P_x \longrightarrow \mathbb{P}V_x$.

Proposition 4.1 *The map $\phi_x : P_x \longrightarrow I\!\!PV_x$ is the natural map*

$$P_x \longrightarrow \mathcal{K}(P_x) \subset I\!\!P(H^0(P_x, 2\theta_{P_x}) \simeq I\!\!PV_x.$$

Proof: : See [5], Proposition 1, p.745.

Proposition 4.2 *For any curve C and any x in $X[2]-\{0\}$, we have $\mathcal{K}(C) \cap I\!\!PV_x = \mathcal{K}(P_x[2])$, (the Schottky Jung relations).*

Proof: : See [5], Proposition 2 (1), p.746.

5 Situation in $I\!\!P(H^0(L))$, when $g = 3$.

We now assume $B = J(C)$, where $J(C)$ is the Jacobian of a non-hyperelliptic curve C of genus 3. (This is the generic situation, since the dimension of the moduli space of principally polarized abelian threefolds is 6 which equals the dimension of the moduli space of curves of genus 3.) Recall the results of Narasimhan and Ramanan ($Theorem 1.3, Theorem 1.4$), to obtain a morphism

$$J(C) \stackrel{\phi_{2\theta}}{\longrightarrow} \mathcal{K}(C) \subset F \subset |2\theta|$$

where

1) F is a quartic hypersurface and is the isomorphic image of the moduli space $SU_C(2)$ and

2) the Kummer variety $\mathcal{K}(C)$ is precisely the singular locus of F.

We will use the following

Proposition 5.1 *Let L be an ample line bundle of type $\delta = (d_1, d_2, ..., d_g)$ on an abelian variety A. Then the set of irreducible representations of the theta group $\mathcal{G}(L)$, where $\alpha \in \mathbb{C}^*$ acts as multiplication by α^n(called as of 'weight n'), is in bijection with the set of characters on the subgroup of $n-$torsion elements, $K(L)_n$, of $K(L)$. Moreover, the dimension of any such representation is $\frac{d_1.d_2...d_g}{(n,d_1)...(n,d_g)}$. ((n, d_i) denotes the greatest common divisor of n and d_i.)*

Proof: : When $n = 2$, the statement is proved in [6], Proposition 3.2, p.142. The same proof holds when $n > 2$, by choosing a section over the subgroup of n-torsion elements, $K(L)_n$, of $K(L)$ in the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L) \longrightarrow K(L) \longrightarrow 0$$

in the proof of [6], Proposition 3.2. □

Corollary 5.2 *The quartic F in $|2\theta|$ is $\mathcal{G}(2\theta)$-invariant and the linear span of the eight cubics $\{\frac{dF}{dt_i}\}$ for $i = 0, 1, ..., 7$ form an irreducible $\mathcal{G}(2\theta)$-module where $\alpha \in \mathbb{C}^*$ acts as multiplication by α^3.*

Proof: : Consider the multiplication maps $Sym^n H^0(2\theta) \xrightarrow{\rho_n} H^0(2n\theta)$. Then $I_n = Ker(\rho_n) =$ vector space of degree n forms containing the image $\mathcal{K}(B)$ in $\mathbb{P}H^0(2\theta)$. Since the vector spaces $Sym^n H^0(2\theta)$ and $H^0(2n\theta)$ (via the homomorphism $\mathcal{G}(2\theta) \xrightarrow{\epsilon_n} \mathcal{G}(2n\theta)$) are $\mathcal{G}(2\theta)$-modules, of weight n and ρ_n is equivariant for the $\mathcal{G}(2\theta)$-action, I_n is also a $\mathcal{G}(2\theta)$-module of weight n. Now the homogenous polynomial $F \in I_4$ and the partial derivatives $\frac{dF}{dt_i} \in I_3$. By 5.1, it follows that F is $\mathcal{G}(2\theta)$-invariant , upto scalars. If $z \in \mathcal{G}(2\theta)$, then $z\frac{dF}{dt_i} = \frac{d(zF)}{d(zt_i)} = \alpha\frac{dF}{d(zt_i)} \in W = \mathbb{C}\{\frac{dF}{dt_i}\}_{i=0}^{7}$, for some scalar α. Hence W is a $\mathcal{G}(2\theta)$-module of weight 3. By 5.1, dimension of such an irreducible representation is 8. This proves our assertion. □

Similarly, we see that $R = F(s_0^2, ..., s_7^2)$ is a $\mathcal{G}(L)$-invariant octic hypersurface in $\mathbb{P}H^0(L)$, by applying 5.1.

Recall the Weil form e^L on $K(L)$ and the isotropic subgroup $G =< \sigma_1, \tau_1^2, \tau_2^2 >\subset K(L)$. Then $e^L(\sigma_2 + g, \sigma_1) = -1$, for all $g \in G$. Let $a = \sigma_2 + g$, for $g \in G$ and $a' = \sigma_2' + g' \in \mathcal{G}(L)$.

Recall the basis $\{s_0, s_1, ..., s_7\}$ of $H^0(L)$ and $\{s_0^2, ..., s_7^2\}$ of $H^0(M^2)$, (see Section 3). Let W_a^+ and W_a^- denote the eigen spaces in $H^0(L)$, for the action of a'. Now $\mathbb{P}W_a^{\pm} = \{s = 0 : s \in W_a^{\mp}\}$ and $\mathbb{P}V_a^+ = \{t = 0 : t \in H^0(M^2)_a^-\}$. Now $W_{\sigma_2}^{\pm} = \mathbb{C}\{s_0 \pm s_1, s_2 \pm s_4, s_3 \pm s_5, s_6 \pm s_7\}$ and $H^0(M^2)_{\sigma_2}^- = \mathbb{C}\{s_0^2 - s_1^2, s_2^3 - s_4^2, s_3^2 - s_5^2, s_6^2 - s_7^2\}$.

Then p restricts on $\mathbb{P}W_{\sigma_2}^{\pm} \longrightarrow \mathbb{P}V_{\sigma_2}^+$ as $(s_0; s_2 : s_3, s_6) \mapsto (s_0^2 : s_2^2 : s_3^2 : s_6^2)$, of degree 2^3. Similarly, one checks that if $a = \sigma_2 + g, g \in G$ then p restricts to $\mathbb{P}W_a^{\pm} \longrightarrow \mathbb{P}V_{\sigma_2}^+$ as $(z_0 : ... : z_3) \mapsto (z_0^2 : ... : z_3^2)$ of degree 2^3.

Proposition 5.3 *Consider a principally polarized abelian surface (Y, P), which is not a product of elliptic curves. Let $y_1, y_2 \in Y$ be elements of order 2, such that $e^{P^2}(y_1, y_2) = -1$. Then we have the following.*

1) There is a polarized abelian surface (Z, N), such that N is strongly symmetric of type $(1, 4)$ and there is a covering map $f : Z \longrightarrow Y$ with the Galois group of the map f being isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$.

2) The vector space $H^0(N)$ *can be written as*

$$H^0(N) = H^0(P) \oplus H^0(t^*_{y_1}P) \oplus H^0(t^*_{y_2}P) \oplus H^0(t^*_{y_1+y_2}P).$$

and there is a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\phi_N} & \phi_N(Z) & \subset & \mathbb{P}^3 = \mathbb{P}H^0(N) \\ \downarrow f & & \downarrow & & \downarrow q \\ Y & \xrightarrow{\phi_{P^2}} & \mathcal{K}(Y) & \subset & \mathbb{P}^3 = \mathbb{P}H^0(M^2) \end{array}$$

where $q(r_0 : r_1 : r_2 : r_3) = (r_0^2 : r_1^2 : r_2^2 : r_3^2)$*. Here* $\{r_0, r_1, r_2, r_3\}$ *is a basis obtained from above decomposition of* $H^0(N)$*, such that* $r_0, r_1, r_3 \in H^0(N)^+_i$ *and* $r_3 \in H^0(N)^-_i$*.*

Proof: : 1) Consider the isomorphism $\phi_P : Y \longrightarrow Pic^0(Y)$, $b \mapsto t_b^*P \otimes P^{-1}$. Let L_{y_1} and L_{y_2} denote the images of y_1 and y_2 under this map. These two line bundles define an unramified cover, $f : Z \longrightarrow Y$, whose Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as asserted.

Then $N = f^*P$ is an ample line bundle and $dim H^0(N) = 4$. So to see that N is of type $(1,4)$, it is enough to show that $K(N)$ has an element of order 4. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\psi_N} & Pic^0(Z) \\ \downarrow f & & \uparrow \hat{f} \\ Y & \xrightarrow{\psi_M} & Pic^0(Y) \end{array}$$

Then $\hat{f} \circ \psi_M(y_i) = 0$. This implies that if z_1 and z_2 are in Z such that $f(z_i) = y_i$, then $z_1, z_2 \in K(N)$. Moreover, since $e^{P^2}(y_1, y_2) = -1$ and $N^2 \simeq f^*(P^2)$, we have $e^{N^2}(z_1, z_2) = -1$. This gives $e^N(z_1, z_2) = \pm i$. Hence the elements $z_1, z_2 \in K(N)$ are of order 4.

2) Clearly, $f_*N = P \oplus (P \otimes L_{y_1}) \oplus (P \otimes L_{y_2}) \oplus (P \otimes L_{y_1+y_2})$. Now, in the algebraic equivalence class of N, there are strongly symmetric line bundles. Hence, by tensoring P with a suitable line bundle of order 2, we may assume that $N = f^*P$ is strongly symmetric and $r_0 \in H^0(P)$ is such that $i(r_0) = r_0$.

Since N is strongly symmetric, by 3.1, $\delta_{-1}(z_j^{'})^2 = (z_j^{'})^2$, for some lifts $z_j^{'} \in \mathcal{G}(N)$ of $z_j \in K(N)$. We may further choose the lifts such that $\delta_{-1}(z_j^{'}) = (z_j^{'})^{-1}$, (as in 3.2). In particular, the descent data of N to P is $K^{'} =< (z_1^{'})^2, (z_2^{'})^2 > \subset \mathcal{G}(N)$, which is a splitting over $K =< z_i^2, z_2^2 > \subset K(N)$ in the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(N) \longrightarrow K(N) \longrightarrow 0.$$

This means $(z_j^{'})^2 r_0 = r_0$. Also this gives

As in 3.5, we see that

$$i.z_j^{'}(r_0) = z_j^{'}(r_0)$$

and

$$i.z_1^{'}.z_2^{'}(r_0) = -z_1^{'}.z_2^{'}(r_0).$$

Thus $r_0, r_1 = z_1^{'}(r_0), r_2 = z_2^{'}(r_0) \in H^0(N)_i^+$ and $r_3 = z_1^{'}.z_2^{'}(r_0) \in H^0(N)_i^-$.

Hence one sees as earlier that $Gal(q) =< z_1^2, z_2^2, i >$, with a commuatative diagram as in 5.3. □

Proposition 5.4 *Let $a = \sigma_2 + g$, $g \in G$ and $\mathbb{P}W_a$ denote an eigenspace of a in $\mathbb{P}H^0(L)$. Then there is an abelian surface Z and a symmetric line bundle N on Z of type $(1,4)$ such that $Z \xrightarrow{\phi_N} \mathbb{P}(H^0(N)) \overset{Heis(4)}{\simeq} \mathbb{P}W_a \subset \mathbb{P}H^0(L)$. Moreover, under this isomorphism, the image $\phi_N(Z)$ is mapped onto the $Heis(4)$-invariant surface $S = R \cap \mathbb{P}W_a$, where R is the $Heis(2,4)$- invariant hypersurface of degree 8 in $\mathbb{P}H^0(L)$, defined by $F(s_0^2 : s_1^2 : ,,,:, s_7^2) = 0$. (F being the Coble quartic).*

Proof: : Consider the restricted morphism $p : \mathbb{P}W_a \longrightarrow \mathbb{P}V_a$, given as $(z_0 : ... : z_3) \mapsto (z_0^2 : ... : z_3^2)$. Then a acts trivially on $\mathbb{P}W_a$ and $a^{\perp}/a (\simeq Heis(4))$ acts on $\mathbb{P}W_a$, (here $a^{\perp} = \{y \in K(L) : e^L(a,y) = 1\}$). Hence there is a $Heis(4)$- action on $\mathbb{P}W_a$ and similarly a $Heis(2,2)$- action on $\mathbb{P}V_a$. By 4.1, there is a principally polarized abelian surface (P_a, θ_{C_a}), (P_a being the Prym variety associated to the element $\pi(a) \in K(M^2)$), such that

$$P_a \longrightarrow \mathcal{K}(P_a) \subset |2\theta_{C_a}| \simeq \mathbb{P}V_a.$$

Consider the images of τ_1, τ_2, which are elements of order 2 in $J(C)$. Since $e^{L^2}(\tau_i, a) = 1$, for the Weil form $e^{2\theta}$ on $J(C)[2]$, $\pi(\tau_1), \pi(\tau_2) \in \pi(a)^{\perp}/\pi(a)$. Moreover, $e^{2\theta}(\pi(\tau_1), \pi(\tau_2)) = -1$. By 4.2, the points $\phi_{M^2} \circ \pi(\tau_i)$, are nodes in the Kummer of the Prym variety P_a. These nodes correspond to elements of order 2 in P_a, say β_1 and β_2. Since the Weil form $e^{2\theta_{C_a}}$ on $P_a[2]$ is induced from the Weil form $e^{2\theta}$, we have $e^{2\theta_{C_a}}(\beta_1, \beta_2) = -1$. By 5.3, there is a polarized abelian surface (Z, N) of type $(1,4)$, such that the following diagram commutes

$$\begin{array}{ccccc} Z & \xrightarrow{\phi_N} & \phi_N(Z) & \subset & \mathbb{P}H^0(N) \\ \downarrow f & & \downarrow & & \downarrow q \\ P_a & \xrightarrow{\phi_{2\theta_{C_a}}} & \mathcal{K}(P_a) & \subset & |2\theta_{C_a}| \end{array}$$

and for the choice of basis $\{r_0, r_1, r_2, r_3\}$, in 5.3 2), the morphism q is defined as $(r_0 : r_1 : r_2 : r_3) \mapsto (r_0^2 : r_1^2 : r_2^2 : r_3^2)$, with $Gal(q) =< z_1^2, z_2^2, i >$, ($z_j$ as in 5.3).

Now, R is the $Heis(2,4)$-invariant octic $F(s_0^2 : ... : s_7^2) = 0$, where F is the Coble quartic. Note that $S = R \cap I\!PW_a$ is $a^{\perp}/a$-invariant and is mapped onto the Kummer, $K(P_a)$, under the restriction morphism. Moreover, the Galois group of $p_{|S}$ is $< \tau_1^2, \tau_2^2, i >$ which is isomorphic to the Galois group of q. Hence there is a $Heis(4)$- isomorphism $I\!PH^0(N) \longrightarrow I\!PW_a$, such that the Heisenberg invariant octic surface $\phi_N(Z)$ is mapped onto the Heis(4)-invariant octic surface $S = R \cap I\!PW_a$. This proves the assertion. $\square$

It is known that the Kummer $\mathcal{K}(P_a)$, has 6 of its nodes in each of the coordinate hyperplane, namely the coordinate points and 3 other distinct points. The preimages of the coordinate points are the coordinate points in $I\!PH^0(N)$ and q is etale over the other 3 points which are the pinch points of $\phi_N(Z)$ in the respective coordinate hyperplane.

Proposition 5.5 *$\phi_N(Z)$ has exactly 48 pinch points, 12 in each coordinate hyperplane.*

Proof: : See [3], Proposition 2.2, p.633.

Let T_a denote the set of pinch points and the coordinate points in $\phi_N(Z)$.

Proposition 5.6 *The components $h(\phi_L(A)), h \in H$ (here $H = J/(G' \times i)$) and $I\!PW_a$ intersect at the subset T_a of $\phi_N(Z)$. In particular $\cap_{h \in H} h(\phi_L(A)) = \cup_{a=\sigma_2+g, g \in G} T_a$.*

Proof: : Since $\pi^{-1}\mathcal{K}(C) = \cup_{h \in H} h(\phi_L(A))$, by 4.2 and 5.5, we conclude that $h(\phi_L(A)) \cap I\!PW_a = T_a$, for all $h \in H$. This gives the assertion. $\square$

6 Some remarks

a) Consider the moduli space $\mathcal{A}^l_{(1,2,4)}$ of triples $(A, c_1(L), f)$, where $f : K(L) \longrightarrow \boldsymbol{Z}/D\boldsymbol{Z} \times \boldsymbol{Z}/D\boldsymbol{Z}$ is a level structure, (here $D = (1, 2, 4)$). Consider the subset of $\mathcal{A}^l_{(1,2,4)}$, $\mathcal{A}^{lo}_{(1,2,4)}$, parametrizing triples which admit a $(\boldsymbol{Z}/2\boldsymbol{Z})^3-$isogeny to the Jacobian of a non-hyperelliptic curve.

Since $dim\mathcal{A}^{lo}_{(1,2,4)} = dim\mathcal{A}^l_{(1,2,4)} = 6$ and $c_1(L)$ gives a birational morphism , $\mathcal{A}^{lo}_{(1,2,4)}$ is an open subset of $\mathcal{A}^l_{(1,2,4)}$.

Consider a triple $(A, c_1(L), f) \in \mathcal{A}^{lo}_{(1,2,4)}$. We have seen that there is a $Heis(2,4)$-invariant octic hypersurface R, defined by $F(s_0^2 : s_1^2 : ... : s_7^2) = 0$, (F being the Coble quartic), such that $\phi_L(A) \subset R \subset I\!PV(2,4)$. In fact $h(\phi_L(A)) \subset Sing(R)$, for all $h \in H$, (H as in 5.6).

Now F is a $Heis(2,2,2)$-invariant quartic polynomial in $I\!PV(2,2,2)$. Since the space of $Heis(2,2,2)$-invariant quartics is 14-dimensional, (see [4], p.186]), the space of $Heis(2,4)$-invariant octics in $I\!P^7$ which are of the form $R = F(s_0^2 : ... : s_7^2)$ where F is a $Heis(2,2,2)$-invariant quartic, is also 14-dimensional. Call this space as

$P(Sym^8V(2,4)^{Heis(2,4)'}) = I\!P^{14}$.

So there is a morphism

$$\mathcal{A}^{lo}_{(1,2,4)} \xrightarrow{T} I\!P^{14}$$

where T is defined as $(A, c_1(L), f) \mapsto R$.

One may try to study this morphism, from a moduli point of view.

b) Consider the special basis $\{s_0^2, ..., s_7^2\}$ (which is different from the usual *Heisenberg* basis) of $H^0(2\theta)$ and the action of the elements of the subgroup $< \sigma_2, \tau_1^2, \tau_2^2 > \subset K(2\theta)$ on this basis (see 3.8).

Also, by 3.12, the points $b(P_0) \in \phi_L(A)$, where $b \in < \sigma_2, \tau_1, \tau_2 > \subset K(L)$, $P_0 = (0 : ... : 0 : c : 1)$ and the point $Q_0 = (0 : ... : 0 : c^2 : 1) \in \mathcal{K}(C)$, for some non-zero $c \in \mathbb{C}$. With these data, in addition to knowing the geometry of $SU_C(2)$ in $|2\theta|$- linear system one may try to know the equation of the *Coble quartic*, in terms of this basis $\{s_0^2, ..., s_7^2\}$.

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