Proceedings of Symposia in Pure Mathematics
Volume 55 (1994), Part 1

The Standard Conjectures

STEVEN L. KLEIMAN

ABSTRACT. This paper introduces the formalism of Grothendieck's two "standard conjectures". We discuss the context in which the conjectures arose, their implications for the category of numerical equivalence (neq) motives, the way they explain the Weil conjectures, the basic theory of correspondences, eight important forms of the Lefschetz standard conjecture, and finally the Hodge standard conjecture and its implications.

1. Introduction

Grothendieck published only one article [5] about the two conjectures on algebraic cycles, which he called the "standard conjectures". The article is short and expository; it states a number of implications and indicates their significance, but gives no proofs. The proofs, together with further development of the theory, appeared at about the same time in the author's article [12], which was written at Grothendieck's request and with his aid and encouragement (but without his revealing that [5] was in the works). In this paper, we review the old theory and the subsequent developments.

Grothendieck [5, p. 193] wrote that the conjectures "arose from an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties ... and they were worked out about three years ago independently by Bombieri and myself." He concluded his article with these words: "The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called 'theory of motives' which is a systematic theory of 'arithmetic properties' of algebraic varieties as embodied in their groups of classes of cycles for numerical equivalence. ... Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry."

1991 *Mathematics Subject Classification*. 14C25; Secondary 14G10.
Supported in part by National Science Foundation grant 8801743.

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0082-0717/94 $1.00 + $.25 per page

Grothendieck formulated the two conjectures for smooth, projective varieties X over an algebraically closed ground field k using the ℓ-adic étale cohomology groups $H^i(X)$ where ℓ is a prime different from the characteristic of k. He fixed an isomorphism of the ℓ-group of roots of unity in k^* with $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ "for simplicity" so that each algebraic cycle of codimension i has a cohomology class in $H^{2i}(X)$. This act is unusual (within parentheses, Grothendieck called it a "heresy!");[1] however, it does serve to clarify the nature of the conjectures and their consequences. For example, as we shall see, the conjectures imply all three Weil conjectures—indeed, explain them—yet there is no need to keep track of twisting by roots of unity. Furthermore, the theory looks more geometric.

The two conjectures are easy to state. The first, the *Lefschetz standard conjecture*, asserts that an abstract analogue of the Λ-operator of Hodge theory is induced by an algebraic cycle on $X \times X$. This conjecture has other forms, which will be discussed in §4; however, Grothendieck [**5**, bottom of p. 196] wrote that it "seems to be most amenable" in this form. The second conjecture, the *Hodge standard conjecture*, asserts that there is an abstract version of the Hodge index theorem for the $\mathbf{Q}$-vector space of classes of algebraic cycles. In characteristic zero, the second conjecture holds of course, but the first is still unknown.

Given the two standard conjectures, the category of *numerical equivalence* (neq) *motives* has marvelous properties indeed. Grothendieck published nothing on motives himself, but his ideas were explained and developed twenty years ago by Demazure [**2**], Manin [**14**], Saavedra [**15**], and the author [**13**]. See also Scholl's report [**16**, §1] in these proceedings. Scholl explains, among other things, a lovely variant, due to Jannsen [**8**, pp. 447–8], of Grothendieck's construction of the category of neq motives.

Grothendieck's construction of the category of neq motives proceeds in stages. First, the *neq correspondence category* is formed: an object is a smooth, projective variety X; and a set of maps $\operatorname{Hom}(X, Y)$ is the $\mathbf{Q}$-vector space of neq classes of algebraic cycles of codimension r on $X \times Y$ where $r := \dim X$; the composition of a class u on $X \times Y$ with a class v on $Y \times Z$ is the class $p_{13*}(p_{12}^*u \cdot p_{23}^*v)$ where the p's are the projections. The category has direct sums and tensor products; they are induced by the disjoint union and Cartesian product of varieties. Next, the category of *neq effective motives* is formed by formally adding images of all projectors (idempotent endomorphisms) π; the objects of this category are the pairs (X, π). For example, the class of $X \times \{\text{point}\}$ is a projector, and if $X = \mathbf{P}^1$, then the corresponding motive is called the *Lefschetz motive* and denoted by $\mathbf{L}$. Finally, the whole category of neq motives is obtained by formally adding the tensor product inverse $\mathbf{T}$ of $\mathbf{L}$, called the *Tate motive*.

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[1] The exclamation point is Grothendieck's.

The category of neq motives is conjectured to be the subcategory of semisimple objects in a larger category of "mixed motives".[2] Indeed, there should be a more refined local and relative theory of motivic sheaves and motivic cohomology. In this conjectured theory, there is an important role to be played by a host of natural equivalence relations that filter the groups of algebraic cycles and breach the gap between numerical equivalence and rational equivalence. Work in this direction has been carried out by Beilinson, Bloch, Jannsen, Murre, and many others; see Jannsen's comprehensive report [9] in these proceedings.

The standard conjectures imply that each neq motive has a semisimple endomorphism ring, or equivalently, that the category is semisimple abelian. (The basic reason why was given by Weil for curves, and developed by Serre [19] in higher dimensions.) Grothendieck called this fact a "miracle" (the author personally heard him do so on several occasions). However, in a remarkable piece of work, Jannsen [8] recently proved this semisimplicity without assuming the conjectures. In fact, he proved the converse: if an equivalence relation on cycles gives rise in similar fashion to a semisimple category of motives, then the relation is necessarily numerical equivalence. Jannsen's proof is short and elementary; it could have been found in the 1960s when motives were first considered. However, unlike the standard conjectures, Jannsen's work does not yield the semisimplicity of the Frobenius endomorphism, which yields the (E. Artin–Weil) Riemann hypothesis. Nevertheless, perhaps Jannsen's success is a sign that the standard conjectures are more tractable than we have come to believe.

The two standard conjectures have another important consequence, which is unknown even in characteristic zero: the coincidence of homological equivalence and numerical equivalence of algebraic cycles. When this coincidence occurs on $X \times Y$, then every map from the neq motive of X to that of Y induces a map from the cohomology of X to that of Y. Sometimes, the coincidence of the two relations can be proved by sandwiching homological equivalence between numerical equivalence and another equivalence relation. For example, an old theorem asserts that, if a divisor is numerically equivalent to 0, then some multiple is algebraically equivalent to 0, and a fortiori homologically equivalent to 0. At first, it was hoped that a similar converse would hold for cycles of arbitrary codimension, but this hope was dashed in 1969 when Griffiths found a counterexample. Recently, Beilinson speculated (private communication, fall 1989) that algebraic equivalence might be replaced by a broader relation such as this one, which he called "Drinfeld equivalence": a cycle Z on X is Drinfeld equivalent to 0 if there exists

[2] In a letter of March 25, 1992 to the author, Serre wrote: "It is difficult for me to recall my discussions (of 1964–65) with Grothendieck precisely, but I am almost sure that: (a) he dreamed about motives which we now called mixed; I remember for instance telling him that the corresponding ℓ-adic representations are no more semi-simple (the example being an extension of an Abelian variety by a multiplicative group); (b) he had no precise definition of them (that I knew of); this is probably why he did not mention them in print".

a proper, smooth, and connected family F/T, a cycle C on F, and two points s, t of T such that $F(s) = X$ and $C(s) = Z$ and $C(t) = 0$. For example, if Z is algebraically equivalent to 0, then we may take $F = X \times T$ for a suitable T.

The Lefschetz standard conjecture has a weaker form, which asserts that the composition of the projection and the inclusion,

$$H^*(X) \twoheadrightarrow H^i(X) \rightarrowtail H^*(X),$$

is induced by an algebraic cycle π^i on $X \times X$. Consequently, the neq motive of X decomposes into a direct sum of pieces that correspond to the individual groups $H^i(X)$.

The standard conjectures imply therefore that the canonical functor

$$h\colon ((\text{smooth, projective varieties})) \longrightarrow ((\text{neq motives}))$$

is a sort of universal cohomology theory: any contravariant functor that is formally like $X \mapsto H^*(X)$ and satisfies the standard conjectures factors through h. Hence, the category of motives may be used as an abstract substitute for singular cohomology to compare "motivated" properties of the various cohomology theories. For example, every endomorphism of a variety induces an endomorphism of each of its cohomology groups. Does the characteristic polynomial have integer coefficients that are independent of the theory? The answer is yes if the standard conjectures hold; see Corollary 5-5.

As are other universal objects, the category of neq motives is constructed by making the minimal modifications necessary to the original object, the category of smooth, projective varieties. From this point of view, each induced functor

$$\rho\colon ((\text{neq motives})) \longrightarrow ((\text{graded } \mathbf{Q}\text{-vector spaces}))$$

is called a "realization" of the category of motives. Nowadays, some prefer to start from the category of vector spaces and work toward the category of motives (see [1] for example). From this point of view, the various functors ρ are called "improvements".

Section 2 of this paper introduces the three Weil conjectures and their connection with the standard conjectures. Section 3 introduces the notion of a Weil cohomology and develops the theory of cohomological correspondences, which forms the basis of the theory of the two standard conjectures. Section 4 introduces the cohomology operator Λ and three related operators. Then it states eight forms of the Lefschetz standard conjecture and investigates the relationship among them. Section 5 states the Hodge standard conjecture. Then it shows that, in the presence of this conjecture, the Lefschetz standard conjecture is equivalent to another conjecture, that homological equivalence and numerical equivalence coincide. Finally, it shows how the two standard conjectures yield the refined form of the third Weil conjecture, the Riemann hypothesis, which asserts this: the action of the Frobenius endomorphism Φ on the cohomology group $H^i(X)$ is semisimple, its characteristic polynomial has integer coefficients, which are independent of the choice of cohomology theory, and its eigenvalues are of absolute value $q^{i/2}$. The proofs in this paper on occasion refer to [12] for details; however, the spirit of each proof normally remains intact.

2. The Weil conjectures

Let X be an irreducible, smooth, projective variety of dimension r defined by polynomials with coefficients in the finite field $\mathbf{F}_q$ with q elements, where q is a power of the characteristic p. For each integer $n \geq 1$, let ν_n denote the number of points of X with coordinates in the extension field $\mathbf{F}_{q^n}$; in other words,

$$\nu_n := \#X(\mathbf{F}_{q^n}).$$

Form the following "logarithmic" generating function for ν_n:

$$\log Z(t) := \sum_{n\geq 1} \nu_n t^n / n.$$

The function $Z(t)$ is called the *zeta function* of X, although another function $Z(q^{-s})$ is denoted by $\zeta(s)$.

By specifying the shape of $Z(t)$, the Weil conjectures describe the growth of ν_n. The function $Z(t)$ was introduced in 1923 by E. Artin. He did so after Hecke had revived interest in Dedekind's extension of Riemann's zeta function to number fields. Artin considered hyperelliptic function fields over $\mathbf{F}_q$ and proved that $Z(t)$ is a rational function with a functional equation. He conjectured that its zeros lie on the circle $|t| = q^{1/2}$ and called this conjecture the "Riemann hypothesis". In 1931, F. K. Schmidt reformulated the theory in the language of algebraic geometry and extended the formulation to arbitrary smooth, projective curves. Weil formulated his conjectures in arbitrary dimension around 1950. Some additional history is given below; to learn more, see Dieudonné's fascinating article [3], his extensive book [4], and Katz's masterful introduction [10].

For example, consider the projective r-space $\mathbf{P}^r$. The standard decomposition,

$$\mathbf{P}^r = \mathbf{A}^0 \amalg \mathbf{A}^1 \amalg \cdots \amalg \mathbf{A}^r,$$

of $\mathbf{P}^r$ into the disjoint union of affine spaces yields

$$\nu_n = 1 + q^n + \cdots + q^{nr}.$$

So the definition of $\log Z(t)$ yields

$$\log Z(t) = \sum t^n/n + \sum q^n t^n/n + \cdots + \sum q^{nr} t^n/n.$$

Now, use the standard power series expansion,

$$\log(1-u) = -\sum u^n/n.$$

It yields the final expression for $Z(t)$ as a rational function,

$$Z(t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^r t)}.$$

To treat a general X, we use its Frobenius endomorphism Φ, which carries a point x of X to the point x^q of X, whose coordinates are the qth powers of those of x:

$$\Phi: X \to X, \qquad \Phi(x) := x^q.$$

Obviously, ν_n is equal to the number of points x of X left fixed by the nth iterate Φ^n, at least set-theoretically. In fact, ν_n is equal to the weighted number of fixed points; for the latter is, by definition, the intersection number on $X \times X$ of the graph of Φ^n with the diagonal, and the intersection is transverse. Hasse introduced this use of Φ in 1936, and then he and Deuring pointed out the relevance of the theory of correspondences.

Finally, let $H^i(X)$ be the ith ℓ-adic cohomology group as in §1. Then the *Lefschetz fixed-point formula* (or *trace formula*) expresses ν_n as the alternating sum of the traces of the endomorphisms of the $H^i(X)$ induced by pullback under Φ^n:

$$\nu_n = \sum_{i=0}^{2r} (-1)^i \operatorname{Tr}(\Phi^n | H^i(X)).$$

The groups $H^0(X)$ and $H^{2r}(X)$ are 1-dimensional vector spaces and, on them, Φ^n induces the identity and multiplication by q^{rn} respectively, because $\Phi^n: X \to X$ is a finite surjective map of degree rn. For $0 < i < 2r$, let w_{ij} be the eigenvalues of $\Phi | H^i(X)$. Then the eigenvalues of $\Phi^n | H^i(X)$ are the nth powers w_{ij}^n. Hence,

$$\nu_n = 1 + \sum_{i=1}^{2r-1} (-1)^i \sum_j w_{ij}^n + q^{rn}.$$

As in the example of $\mathbf{P}^r$ above, using the expansion of $\log(1-u)$, we find that $Z(t)$ is a rational function

$$Z(t) = \frac{P_1(t) \cdots P_{2r-1}(t)}{P_0(t) P_2(t) \cdots P_{2r-2}(t) P_{2r}(t)},$$

where the $P_i(t)$ are not exactly the characteristic polynomials, but

$$P_i(t) := \prod_j (1 - w_{ij} t) = \det((1 - t\Phi) | H^i(X)).$$

Moreover, $P_0 = 1 - t$ and $P_{2r}(t) = 1 - q^r t$.

The rationality of $Z(t)$ is the first of the three Weil conjectures; in fact, Weil himself explained the above way of using the Lefschetz formula.[3] However, the first proof of the rationality was given in 1960 by Dwork, who, instead, made an ingenious use of p-adic analytic functions; moreover, Dwork

[3] In a letter of March 25, 1992 to the author, Serre wrote: " ... the idea of counting points over $\mathbf{F}_q$ by a Lefschetz formula is entirely *an idea of Weil.* I remember how enthusiastic I was when he explained it to me, and a few years later I managed to convey my enthusiasm to Grothendieck (whose taste was not a priori directed towards finite fields)."

proved the rationality for an *arbitrary* variety, one that need not be smooth or projective. In 1963, M. Artin and Grothendieck developed enough of the theory of étale cohomology to justify the use of Weil's Lefschetz formalism for smooth, projective varieties. Furthermore, they proved a more general result, which Weil had conjectured as well, but which Dwork's methods did not yield: the rationality of certain L-functions, generalizing those introduced by E. Artin. In addition, they proved a base change theorem and a comparison theorem, which imply another part of what Weil conjectured: if X is the reduction mod p of a complex variety X' defined by equations with coefficients in a number field, then $P_i(t)$ has degree equal to the ith Betti number of X' (no w_{ij} vanishes because $\Phi|H^i(X)$ is nonsingular).

Two years later, in 1965, M. Artin, Grothendieck, and Verdier proved the rationality of L-functions of an even more general sort on an arbitrary variety, recovering Dwork's theorem in particular. To do so, they developed a theory of cohomology with compact supports and reduced the general statement to the case in which X is a smooth, projective curve. Finally, in that case, they proved a suitable version of the Lefschetz formula. However, there was still no way to rule out cancellation among the $P_i(t)$ above; so it was still conceivable that the coefficients of the $P_i(t)$ were not ordinary integers and depended on ℓ. Cancellation was ruled out in 1973 by Deligne, for it is ruled out by the Riemann hypothesis.

Poincaré duality yields the second Weil conjecture, which asserts that $Z(t)$ satisfies the following functional equation:

$$Z(1/q^n t) = (-1)^{\chi+\mu} q^{n\chi/2} t^{\chi} Z(t),$$

where χ is the Euler characteristic, the alternating sum of the dimensions of the $H^i(X)$, and where μ is 0 if r is odd, and μ is the multiplicity of $-q^{r/2}$ as an eigenvalue w_{rj} if r is even. (This μ is, unfortunately, missing from [**12**, 4.4, p. 385] as N. Katz kindly pointed out March 4, 1969.) Indeed, under the duality, the transpose of $\Phi|H^{2r-i}(X)$ is equal to $q^r(\Phi|H^i(X))^{-1}$. Hence, up to order, the numbers $w_{(2r-i)j}$ and $q^r w_{ij}^{-1}$ are equal. The functional equation follows via a simple computation. Poincaré duality and the functional equation were also proved in 1963 by Artin and Grothendieck.

The third and last Weil conjecture, the *Riemann hypothesis*, specifies the absolute value of the eigenvalues w_{ij}:

$$|w_{ij}| = q^{i/2}.$$

The w_{ij} are algebraic integers, and each appears along with all its conjugates, because the characteristic polynomial of $\Phi|H^i(X)$ is equal to $t^{b_i}P(t^{-1})$ where $b_i := \dim H^i(X)$, the ith Betti number of X. The conjecture was proved in two different ways in 1933 and 1934 for elliptic curves by Hasse, and in two different ways over the course of the 1940s for curves of arbitrary genus by Weil. The conjecture was finally proved in 1973 for arbitrary X by Deligne.

It is also generally conjectured that the endomorphisms $\Phi|H^i(X)$ are semisimple. This conjecture was proved by Weil for curves, abelian varieties, and a few other varieties, but it is still unknown in general. It is implied by the standard conjectures; see Theorem 5-6.

3. Correspondences

The theory of the standard conjectures is purely formal, so we shall develop it using an arbitrary *Weil cohomology theory.* This is a contravariant functor $X \mapsto H^*(X)$ from the category of *irreducible,* smooth, projective varieties X over an algebraically closed field to the category of graded anticommutative algebras over a "coefficient field" K of characteristic zero, with the following properties:

(1) (finiteness) Each $H^i(X)$ has finite dimension, and vanishes unless $0 \leq i \leq 2r$ where $r = \dim X$.

(2) (Poincaré duality) For each X of dimension r, there is a functorial "orientation" isomorphism $H^{2r}(X) \xrightarrow{\sim} K$ and, preceded by the cup product (multiplication) pairing, it yields a nondegenerate bilinear pairing,

$$H^i(X) \times H^{2r-i}(X) \to K \quad \text{by } x, y \mapsto \langle x \cdot y \rangle,$$

where, for any u in $H^*(X)$, the symbol $\langle u \rangle$ denotes the image under the orientation map of the projection of u in $H^{2r}(X)$. For convenience, given a Y of dimension s and a map $f\colon X \to Y$, let

$$f_*\colon H^i(X) \to H^{2s-2r+i}(Y)$$

denote the transpose of $f^*\colon H^{2r-i}(Y) \to H^{2r-i}(X)$.

(3) (Künneth formula) For each X and Y, the projections induce an isomorphism

$$H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y).$$

(4) (cycle map) For each X, let $C^i(X)$ denote the group of algebraic cycles of codimension i. Then there is a group homomorphism

$$\gamma_X\colon C^i(X) \to H^{2i}(X),$$

called the "cycle map", satisfying

(i) (functoriality) for each map $f\colon X \to Y$,

$$f^*\gamma_Y = \gamma_X f^* \quad \text{and} \quad f_*\gamma_X = \gamma_Y f_*,$$

(ii) (multiplicitivity) $\gamma_{X \times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$, and

(iii) (calibration) if P is a point, then $\gamma_P\colon C^0(P) \to H^0(P)$ is equal to the canonical inclusion of the integers $\mathbf{Z}$ into the coefficient field K.

(5) (weak Lefschetz theorem) Let $h\colon W \to X$ be the inclusion of a smooth hyperplane section, and set $r := \dim X$. Then the induced map $h^*\colon H^i(X) \to H^i(W)$ is an isomorphism for $i \le r-2$ and an injection for $i = r-1$.

(6) (strong Lefschetz theorem) Let W be a smooth hyperplane section of X, set $r := \dim X$, and define the Lefschetz operator

$$L\colon H^i(X) \to H^{i+2}(X) \quad \text{by } Lx := x \cdot \gamma_X(W).$$

Then, for $i \le r$, the $(r-i)$th iterate of L is an isomorphism

$$L^{r-i}\colon H^i(X) \xrightarrow{\sim} H^{2r-i}(X).$$

All of these properties were proved in 1963 for étale cohomology by Artin and Grothendieck, except for the last one, the strong Lefschetz theorem. It was proved in 1973 by Deligne at the same time that he proved the Riemann hypothesis. To everyone's surprise, the Lefschetz theorem turned out to be the deeper result. Immediately afterwards, Katz and Messing [11] proved that, because the strong Lefschetz theorem holds for étale cohomology, it holds, when the ground field is the algebraic closure of a finite field, for any cohomology theory, like crystalline cohomology, that possesses all the other properties, except possibly (4), the existence of a cycle map.

The properties above imply that the cycle map γ_X preserves product; indeed, if $\delta\colon X \to X \times X$ is the diagonal map, then

$$\begin{aligned}\gamma_X(Z \cdot W) &= \gamma_X \delta^*(Z \times W) = \delta^*(\gamma_X(Z) \otimes \gamma_X(W))\\ &= \gamma_X(Z) \cdot \gamma_X(W)\end{aligned}$$

for any two properly intersecting algebraic cycles Z and W on X. It is also easy to prove (see [12, 1.2.1, p. 363]) that if some nonzero multiple mZ is algebraically equivalent to 0, then $\gamma_X(Z) = 0$. Denote the $\mathbf{Q}$-vector subspace of $H^{2i}(X)$ generated by the various $\gamma_X(Z)$ by $A^i(X)$.

The Künneth formula, Poincaré duality, and some linear algebra yield the following three canonical isomorphisms:

$$\begin{aligned}H^*(X \times Y) = H^*(X) \otimes H^*(Y) &= \mathrm{Hom}(H^*(X), K) \otimes H^*(Y)\\ &= \mathrm{Hom}(H^*(X), H^*(Y)).\end{aligned}$$

Thus an element u of $H^*(X \times Y)$ may be viewed as a linear map, or operator, from $H^*(X)$ to $H^*(Y)$. Viewed this way, u is called a *correspondence.* If u is in the $\mathbf{Q}$-vector subspace $A^*(X \times X)$, then u is called *algebraic.*

It is easy to see that, if $u = a \otimes b$ in $H^*(X \times Y)$, then $u(x) = \langle x \cdot a \rangle b$. Hence, an arbitrary u is given by the formula

$$u(x) = p_{2*}(p_1^* x \cdot u),$$

where p_1 and p_2 are the projections. Moreover, if u is in $H^{2r+d}(X \times X)$ where $r = \dim X$, then u is equal to a homogeneous linear map of degree

d. On the other hand, if v is in $H^*(Y \times Z)$, then the composition vu of linear maps is identified with a cycle in $H^*(X \times Z)$, namely,

$$vu = p_{13*}(p_{12}^* u \cdot p_{23}^* v),$$

where again the p's are the projections. In particular, this formula shows that if u and v are defined by algebraic cycles, then so is their composition vu.

Here are three basic examples of correspondences; they and others are discussed in more detail in [**12**, pp. 365–6]. First, given a map $g: Y \to X$, let u in $H^*(X \times Y)$ be the class of its graph, and ${}^t u$ in $H^*(X \times Y)$ the "transpose" of u. Then $u = g^*$ and ${}^t u = g_*$. Second, given x in $H^*(X)$, let $u: H^*(X) \to H^*(X)$ be the map of right multiplication by x. Then $u = \delta_* x$ where $\delta: X \to X \times X$ is the diagonal map. In particular, u is equal to its own transpose ${}^t u$. Third, consider the diagonal subvariety Δ of $X \times X$. For $i = 1, \ldots, 2r$ where $r = \dim X$, form the Künneth components of $\gamma_{X \times X}\Delta$:

$$\pi^i \in H^{2r-i}(X) \otimes H^i(X).$$

Then π^i is equal to the composition

$$\pi^i : H^*(X) \twoheadrightarrow H^i(X) \rightarrowtail H^*(X)$$

of the canonical projection and the canonical inclusion. In particular, π^i is a projector. Moreover, obviously, $\pi^{2r-i} = {}^t\pi^i$. So π^{2r-i} is algebraic if and only if π^i is.

A correspondence u in $H^{2r}(X \times X)$ induces an endomorphism of $H^i(X)$ for each i, and the endomorphism's trace is given by the following lovely formula:

(trace formula) $$\mathrm{Tr}(u|H^i(X)) = (-1)^i \langle u \cdot \pi^{2r-i} \rangle.$$

This formula is simple to check; see [**12**, 1.3.6(ii), p. 366], where a more general version is treated as well.

The next result is particularly important because, when combined with the first example above, it implies this: given $f: X \to X$, the induced endomorphism $f|H^i(X)$ is such that its characteristic polynomial has integer coefficients.

THEOREM 3-1. *Assume π^{2r-i} is algebraic where $r = \dim X$. Let u be a correspondence defined by an algebraic cycle on $X \times X$, and let t be a variable. Then $\det((1 - ut)|H^i(X))$ is a polynomial with integer coefficients, and these coefficients are given by universal polynomials in the rational numbers,*

$$s_n := \langle u^n \cdot \pi^{2r-i} \rangle,$$

for $n = 1, \ldots, b_i$ where $b_i := \dim H^i(X)$.

Indeed, by hypothesis, there is an integer m such that $m\pi^{2r-i}$ is defined by an algebraic cycle. Moreover, since u is defined by an algebraic cycle, so is the composition u^n for all $n \geq 0$. Hence ms_n is an integer because γ converts intersection product into "cup" product. Now, the trace formula implies that s_n is equal to the sum of the nth powers of the eigenvalues of $u|H^i(X)$. It follows that the eigenvalues are algebraic integers; see [**12**, 2.8, p. 371]. Hence, the coefficients of the characteristic polynomial are algebraic integers too. However, they are also rational numbers because, solving the Newton identities, we can express them as universal polynomials with rational coefficients in s_n for $n = 1, \dots, b_i$. Thus the proof is complete.

4. The Lefschetz standard conjecture

Fix a Weil cohomology theory $X \mapsto H^*(X)$. The Lefschetz standard conjecture has numerous forms. The most important form involves a natural quasi-inverse (one-sided inverse) Λ to the Lefschetz operator L. We define Λ on $H^i(X)$ for $0 \leq i \leq r$ where $r := \dim X$ by the following commutative diagram:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{r-i}} & H^{2r-i}(X) \\ \Lambda\big\downarrow & & L\big\downarrow \\ H^{i-2}(X) & \xrightarrow{L^{r-i+2}} & H^{2r-i+2}(X) \end{array}$$

in which the two horizontal maps are isomorphisms by the strong Lefschetz theorem. We define Λ on $H^{2r-i+2}(X)$ by the following similar commutative diagram:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{r-i}} & H^{2r-i}(X) \\ L\big\uparrow & & \Lambda\big\uparrow \\ H^{i-2}(X) & \xrightarrow{L^{r-i+2}} & H^{2r-i+2}(X) \end{array}$$

Clearly, Λ is surjective on $H^i(X)$ and injective on $H^{2r-i+2}(X)$.

Alternatively, we can define Λ using *primitive elements*. These are the elements of the following vector space:

$$P^i(X) := \operatorname{Ker}(L|H^i(X)).$$

Clearly $P^i(X)$ is a direct summand of $H^i(X)$, the other summand being $LH^{i-2}(X)$. Hence, each x in $H^i(X)$ has a unique decomposition of the following form, known as its *primitive decomposition*:

$$x = \sum_{j \geq \max(i-r,0)} L^j x_j \quad \text{where } x_j \in P^{i-2j}(X).$$

We can now define Λ by the following formula:

$$\Lambda x := \sum_{j \geq \max(i-r,1)} L^{j-1} x_j.$$

Similarly, we can define three additional useful operators as follows:

$$
\begin{aligned}
{}^c\Lambda x &:= \sum_{j\geq \max(i-r,\,1)} j(n-i+j+1)L^{j-1}x_j;\\
*x &:= \sum_{j\geq \max(i-r,\,0)} (-1)^{(i-2j)(i-2j+1)/2}L^{r-i+j}x_j;\\
p^j x &:= \delta_{ij}x_m \quad \text{where } m := \max(i-r,\,0) \text{ for } j=0,\dots,2r.
\end{aligned}
$$

(The formula for p^j corrects [12, 1.4.2.4, p. 367].) Thus p^j is the projector onto $P^j(X)$ for $j=0,\dots,r$ and $p^j = p^{2r-j}\Lambda^{r-j}$ for $j=r,\dots,2r$. It is easy to check that $*^2=1$ and $\Lambda = *L*$.

We can now state eight forms of the Lefschetz standard conjecture; they and a few more are discussed in greater detail in [12]. Four of the eight simply assert that the above four operators are algebraic. Three of the remaining four sound weaker, but, in fact, six of the seven are outright equivalent, and the seventh is practically equivalent. The eighth form (stated third) is, doubtless, truly weaker. The three principal forms are the following statements:

$A(X, L)$: The restriction $L^{r-2i}\colon A^i(X) \to A^{r-i}(X)$ is an isomorphism for all i.

$B(X)$: The operator Λ is algebraic.

$C(X)$: The projector π^i is algebraic for $0 \leq i \leq 2r$.

The five additional forms of the conjecture are as follows:

${}^cB(X)$: The operator ${}^c\Lambda$ is algebraic.

$\theta(X)$: For each $i \leq r$, there exists an algebraic correspondence θ^i inducing the isomorphism $H^{2r-i}(X) \xrightarrow{\sim} H^i(X)$ inverse to L^{r-i}.

$\nu(X)$: For each $i \leq r$, there exists an algebraic correspondence ν^i inducing an isomorphism $H^{2r-i}(X) \xrightarrow{\sim} H^i(X)$.

${}^pC(X)$: The operator p^i is algebraic for $0 \leq i \leq 2r$.

$*(X)$: The operator $*$ is algebraic.

The following result expresses the relationship among the above eight forms of the conjecture. It also justifies omitting the "L" from $B(X)$.

THEOREM 4-1.
(1) *Conjecture* $A(X\times X, L\otimes 1 + 1\otimes L)$ *implies* $B(X)$.
(2) *Conjecture* $B(X)$ *holds for all choices of* L *if it holds for one.*
(3) *The following conjectures are equivalent:*

$$B(X),\ {}^cB(X),\ \theta(X),\ \nu(X),\ {}^pC(X),\ *(X).$$

(4) *Conjecture* $B(X)$ *implies* $A(X, L)$ *and* $C(X)$.

Indeed, assume $B(X)$, and set $\theta^i := \Lambda^{r-i}$. Then θ^i is algebraic, and it induces an inverse to L^{r-i}. Thus $\theta(X)$ holds.

Assume $\theta(X)$. Clearly, any algebraic correspondence carries $A^*(X)$ into itself. Hence $\theta(X)$ implies $A(X, L)$. Now, the following formula is easy to verify:

$$\pi^i = \theta^i \left(1 - \sum_{j>2r-i} \pi^j\right) L^{r-i} \left(1 - \sum_{j<i} \pi^j\right).$$

Proceeding by induction on i, we conclude that $\theta(X)$ implies $C(X)$. Therefore (4) holds. Finally, $\theta(X)$ implies $B(X)$ because of the following formula, which is easy to verify:

$$\Lambda = \sum_{i\leq r} (\pi^{i-1}\theta^{i+2}L^{r-i+1}\pi^i + \pi^{2r-i}L^{r-i+1}\theta^{i+2}\pi^{2r-i+2}).$$

Trivially, $\theta(X)$ implies $\nu(X)$. Conversely, assume $\nu(X)$ and set $u := \nu^i L^{r-i}$. Then u is algebraic. So, by Theorem 3-1, its characteristic polynomial $P(t)$ has rational coefficients. By the Cayley–Hamilton theorem, $P(u) = 0$. Hence u^{-1} is a linear combination of the powers u^j for $j \geq 0$, and the combining coefficients are rational numbers. So u^{-1} is algebraic. Set $\theta^i := u^{-1}\nu^i$. Then θ is algebraic, and it is the inverse of L^{r-i} on $H^{2r-i}(X)$; in other words, $\theta(X)$ holds.

Since $\nu(X)$ does not involve L, and since $\nu(X)$ is equivalent to $\theta(X)$ and to $B(X)$, the latter two conditions hold for all possible choices of L if either holds for one choice. Thus (2) holds.

Clearly, ${}^cB(X)$ implies $\nu(X)$. Conversely, $\theta(X)$ implies ${}^pC(X)$; in fact, the p^i are given by universal (noncommutative) polynomials with integer coefficients in L and the θ^i, see [12, 1.4.4, p. 368]. Clearly, ${}^pC(X)$ implies ${}^cB(X)$ and $*(X)$. Since $\Lambda = *L*$, obviously $*(X)$ implies $B(X)$. Thus (3) holds.

Finally, assume $A(X \times X, L \otimes 1 + 1 \otimes L)$. Then ${}^c\Lambda \otimes 1 + 1 \otimes {}^c\Lambda$ carries $A^*(X \times X)$ into itself by [12, 1.4.6(ii), p. 368] and [12, 2.1, p. 369]. However, ${}^c\Lambda \otimes 1 + 1 \otimes {}^c\Lambda$ carries the class of the diagonal subvariety Δ into $2{}^c\Lambda$ by [12, 1.3.4, p. 365]. Thus ${}^cB(X)$ holds. So $B(X)$ holds. Thus (1) holds. The proof is now complete.

The final part of the proof is due to Jannsen (private communication, October 24, 1991). Assertion (1) was stated without proof by Grothendieck [5, p. 196]. In [12, 2.13, p. 372], only the following weaker statement was proved: $B(X)$ is implied by $A(X \times X, L \otimes 1 + 1 \otimes L)$ and $B(W)$, where W is a smooth hyperplane section of X. However, the weaker statement is enough to yield the next result.

Corollary 4-2. *Conjecture $A(X, L)$ holds for all X and L if and only if $B(X)$ holds for all X.*

The following result gives some examples of varieties X for which the conjectures are known to hold. It also gives two ways to construct new examples from old ones. Note that, if $H^1(X)$ is the étale cohomology group, then its dimension is equal to twice the dimension of the connected component of the Picard scheme $\mathrm{Pic}^0(X)$.

PROPOSITION 4-3. (1) *Conjecture* $B(X)$ *is stable under product and under hyperplane section.*

(2) *Conjecture* $B(X)$ *holds if* X *is* (a) *a curve,* (b) *a surface such that the dimension of* $H^1(X)$ *is twice that of* $\mathrm{Pic}^0(X)$, (c) *an Abelian variety, or* (d) *a generalized flag manifold* G/P.

(3) *Conjecture* $C(X)$ *holds if* X *is defined by equations with coefficients in a finite field.*

Indeed, $B(X)$ is stable under product because by [12, 1.4.6(ii), p. 368]

$$^c\Lambda_{X\times Y} = {}^c\Lambda_X \otimes 1 + 1 \otimes {}^c\Lambda_Y.$$

If W is a smooth hyperplane section of X, then by [12, 1.4.7(vii), p. 369]

$$\Lambda_W = f^*\Lambda_X^2 f_*;$$

hence, $B(X)$ implies $B(W)$. If X is a curve, then $B(X)$ is trivial. If $X = G/P$, then the algebraic cycles generate $H^*(X)$ because the class of the diagonal is of the form $\sum x_i \otimes y_i$ where x_i and y_i are the classes of algebraic cycles (this argument is in Schubert's 1879 book [17, §39, §41] for $\mathrm{GL}(4, \mathbf{C})$; the argument has been rediscovered several times since then). Since $X \times X$ is equal to $(G\times G)/(P\times P)$, therefore $B(X)$ holds trivially. If X is a surface or an abelian variety, then a few pages of argument are needed to establish $B(X)$; see [12, §2 Appendix, pp. 373–378]. For a surface, the proof is essentially due to Grothendieck; for an abelian variety, the proof grew out of discussions between the author and Lieberman. Finally, (3) was proved by Katz and Messing [11] using the Riemann hypothesis, which Deligne had just established.

COROLLARY 4-4. *The following three conjectures are equivalent:*

$$A(X\times X,\, L\otimes 1 + 1\otimes L),\quad B(X),\quad B(X\times X).$$

Indeed, $A(X\times X, L\otimes 1+1\otimes L)$ implies $B(X)$ by Theorem 4-1 (1). Furthermore, $B(X)$ implies $B(X\times X)$ by Proposition 4-3 (1). Finally, $B(X\times X)$ implies $A(X\times X, L\otimes 1 + 1\otimes L)$ by Theorem 4-1 (4).

5. The Hodge standard conjecture

Fix a Weil cohomology theory $X \mapsto H^*(X)$. The Hodge standard conjecture concerns the cup product pairing on the primitive algebraic cohomology classes on a smooth, projective X of dimension r:

Hdg(X) : For all $i \le r/2$, the $\mathbf{Q}$-valued pairing on $A^i(X) \cap P^{2i}(X)$,

$$x, y \mapsto (-1)^i \langle L^{r-2i} x \cdot y\rangle,$$

is positive definite.

In characteristic zero, the conjecture is true for étale cohomology; indeed, by the Lefschetz principle, we may assume that the ground field is the field of complex numbers, and then the comparison theorem and standard Hodge theory yield $\mathrm{Hdg}(X)$. In arbitrary characteristic, $\mathrm{Hdg}(X)$ holds if X is a surface. A purely algebraic proof, which works in arbitrary characteristic, was given in 1937 by B. Segre [18]; independently, in 1958, Grothendieck [6] gave a similar proof.

There is another widely believed, long-standing conjecture:

$D(X)$: If an algebraic cycle Z on X is numerically equivalent to 0, then $\gamma_X(Z) = 0$; in other words, numerical equivalence and homological equivalence coincide on X.

Of course, if $\gamma_X(Z) = 0$, then Z is numerically equivalent to 0 because γ_X converts intersection product into cup product. Hence, $D(X)$ may be put as follows: On X, homological equivalence of algebraic cycles is the same as numerical equivalence. The relationship between this conjecture and the two standard conjectures is given by the next result.

PROPOSITION 5-1. *Conjecture $D(X)$ implies $A(X, L)$, and the converse holds—in other words, the two conjectures are equivalent—in the presence of* $\mathrm{Hdg}(X)$.

Indeed, assume $A(X, L)$. Clearly, $*$ carries $A^{2r-i}(X)$ into $A^i(X)$. Assume $\mathrm{Hdg}(X)$ too. Then, therefore, the quadratic form on $A^i(X)$,

$$x, y \mapsto \langle x \cdot *y \rangle,$$

is positive definite. Consequently, the canonical pairing

$$A^i(X) \otimes A^{r-i}(X) \to \mathbf{Q} \tag{5-1}$$

is nonsingular. Hence, $D(X)$ holds.

Assume $D(X)$. Then pairing (5-1) is nonsingular. Hence $A^i(X)$ and $A^{r-i}(X)$ have the same dimension, which is finite by the following lemma because $D(X)$ holds. Since the map

$$L^{r-2i}: A^i(X) \to A^{r-i}(X)$$

is injective because of the strong Lefschetz theorem, it is therefore bijective; in other words, $A(X, L)$ holds.

LEMMA 5-2. *Let $C^i_{\mathrm{neq}}(X)$ denote the group of cycles on X modulo numerical equivalence. Then $C^i_{\mathrm{neq}}(X)$ is a free abelian group of finite rank.*

Indeed, form the K-vector subspace of $H^{2r-2i}(X)$ generated by the image of γ_X, and choose $y_1, \dots, y_m$ in the image that form a basis. Consider the map,

$$\alpha: \gamma_X C^i(X) \to \mathbf{Z}^m \quad \text{given by } \alpha(x) := (\langle x \cdot y_1 \rangle, \dots, \langle x \cdot y_m \rangle).$$

It is easy to see that the image of α is equal to $C^i_{\mathrm{neq}}(X)$. So the latter is a free group of finite rank.

COROLLARY 5-3. *Assume that the ground field is the field of complex numbers, that the cohomology theory is the de Rham theory, and that the ordinary Hodge conjecture holds for* X. *Then* $A(X, L)$ *and* $D(X)$ *hold.*

Indeed, it is an immediate consequence of standard Hodge theory that the map

$$L^{r-2p}: H^{p,p}(X) \cap H^{2p}(X, \mathbf{Q}) \to H^{r-p,r-p}(X) \cap H^{2r-2p}(X, \mathbf{Q})$$

is bijective. The ordinary Hodge conjecture asserts that the source and target are equal to $A^i(X)$ and $A^{r-i}(X)$ respectively. Thus $A(X, L)$ holds, and Proposition 5-1 implies that $D(X)$ holds.

PROPOSITION 5-4. *If* $\mathrm{Hdg}(X \times X)$ *and* $B(X)$ *hold, then the operators* Λ, ${}^c\Lambda$, $*$, *and* p^i *and* π^i *for all* i *are defined by algebraic cycles that are independent of the cohomology theory.*

Indeed, $D(X \times X)$ holds by Corollary 4-4, and the π^i are represented by algebraic cycles on $X \times X$ by Theorem 4-1, (3) and (4). These representatives may be chosen without regard to the cohomology theory because their numerical equivalence classes are intrinsically determined as elements of the ring of algebraic correspondences by the following general fact [12, 3.15, p. 382]: a graded, noncommutative ring $E^* = \bigoplus_{\rho=-r}^{r} E^\rho$ with 1 has at most one complete set of orthogonal idempotents $\pi^0, \dots, \pi^{2r}$ such that (a) $E^\rho = \bigoplus_i \pi^{i+\rho} E^* \pi^i$ and (b) for $i = 0, \dots, r$ there exist elements ℓ^i in E^{2r-2i} and λ^i in $E^{-(2r-2i)}$ such that $(\lambda^i \ell^i - 1)\pi^i$ and $(\ell^i \lambda^i - 1)\pi^{2r-i}$ vanish. Now, by Theorem 4-1 (3), the operator ${}^c\Lambda$ is represented by an algebraic cycle; so, see [12, 1.4.6, p. 368], the cycle's numerical equivalence class is uniquely determined by the formula: $[{}^c\Lambda, L] = \sum_{i=0}^{2r} (n-i)\pi^i$. Finally, the remaining operators are given by universal (noncommutative) polynomials in L and ${}^c\Lambda$; see [12, 1.4.3, p. 367, and 1.4.5, p. 368].

The following result gives one reason why the two standard conjectures are important.

COROLLARY 5-5. *Assume* $B(X)$ *and* $\mathrm{Hdg}(X \times X)$.

(1) *Then the Betti numbers* $\dim H^i(X)$ *are independent of the cohomology theory.*

(2) *Let* u *be a correspondence defined by an algebraic cycle on* $X \times X$. *Then its characteristic polynomial has integer coefficients, which are independent of the cohomology theory.*

Indeed, by Proposition 5-4, all the π^i are defined by algebraic cycles that are independent of the cohomology theory. Hence (1) follows from the trace formula in §3, applied to $u := 1_X$. Moreover, (2) follows from Theorem 3-1.

The final result addresses the issue of semisimplicity and the standard conjectures. Connections among Tate's conjecture, semisimplicity, and Conjecture $D(X)$ were explored recently by Deligne, by Jannsen, and by Katz and

Messing; their work appeared in informally distributed handwritten notes of July 1991, and it was incorporated in Tate's article **[20]** in these proceedings.

THEOREM 5-6. *Assume* $B(X)$ *and* $\mathrm{Hdg}(X \times X)$.

(1) *Then, under composition of correspondences,* $A^*(X \times X)$ *is a semi-simple* $\mathbf{Q}$*-algebra.*

(2) (generalized Riemann hypothesis) *Assume* X *is defined by equations with coefficients in the finite field with* q *elements, and let* ϕ *denote its Frobenius endomorphism. Then the induced endomorphism* $\Phi|H^i(X)$ *is semi-simple, its characteristic polynomial has integer coefficients, which are independent of the choice of cohomology theory, and its eigenvalues are of absolute value* $q^{i/2}$.

Indeed, given any correspondence u, set $u' := *{}^t u *$ where ${}^t u$ is the transpose of u. Suppose u is algebraic. Then so is u', because $*(X)$ holds by Theorem 4-1 (3). Now, $C(X)$ holds by Theorem 4-1 (3); hence, the trace formula in §3 implies that $\mathrm{Tr}(u'u)$ is in $\mathbf{Q}$. Furthermore, a calculation shows that $\mathrm{Hdg}(X \times X)$ implies that $\mathrm{Tr}(u'u) > 0$ if $u \neq 0$; see [12, 3.11, p. 381].

To prove (1), suppose u is a nonzero element of the radical. Then $u'u$ is nilpotent, but $u'u \neq 0$ since $\mathrm{Tr}(u'u) \neq 0$. Say $(u'u)^{2^m} = 0$, but $v := (u'u)^{2^{m-1}} \neq 0$. Then $v'v = v^2 = 0$, but $\mathrm{Tr}(v'v) \neq 0$, a contradiction. Thus (1) holds.

Consider (2). By Corollary 5-4, the characteristic polynomial of $\Phi|H^i(X)$ has integer coefficients, which are intrinsic. Finally, set $\Phi_i := \Phi|H^i(X)$ and $g := \sum_i \Phi_i/q^{i/2}$. Then g is an automorphism of the algebra $H^*(X)$ and $g|H^{2r}(X) = 1$. It follows formally [12, 4.2, p. 384] that $g^{-1} = {}^t g$. Clearly, g carries the class of a hyperplane section into itself. Hence, g induces an automorphism of each primitive subspace $P^i(X)$. Therefore, ${}^t g$ commutes with $*$. Hence $g' = *{}^t g * = {}^t g$. By the preceding paragraph, the pairing

$$u,\, v \mapsto \mathrm{Tr}(u'v)$$

is an inner product on the $\mathbf{Q}(q^{1/2})$-algebra generated by g. Since $g'g = 1$, left translation by g preserves this inner product. It follows that g is semi-simple and its eigenvalues have absolute value 1. (The final argument is found in Serre's paper **[19]**.) The proof is now complete.

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Massachusetts Institute of Technology, Cambridge, MA
E-mail address: Kleiman@math.MIT.edu