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The Serre-Swan Theorem for Ringed Spaces

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Plan of the Talk

- Introduction.
- Main Theorem and some special consequences.
- Some examples.

Introduction

- The Serre-Swan Theorem is about the categorical equivalence of vector bundles and that of finitely generated projective modules.
- J. P. Serre in 1955 proved this equivalence in the case of affine schemes.
- In 1962 R. G. Swan proved the same equivalence for topological manifolds.
- In this talk we will try to formulate the Serre-Swan Theorem in the frame work of ringed spaces.

Preliminaries

- **Definition.** A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of ringed on X. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$, such that for all $x \in X$, $\mathcal{O}_{X,x}$ is a local ring.
- For a locally ringed space $(X, \mathcal{O}_X)$, let $\mathfrak{m}_{X,x}$ denotes the unique maximal ideal of $\mathcal{O}_{X,x}$. We will denote by $k(x)$ the *residue field* of X at x, $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.
- We will denote by $\mathcal{O}_X$-**mod** the category of $\mathcal{O}_X$-modules on X.
- For any ring A, let A-**mod** denote the category of A-modules.
- For ringed space $(X, \mathcal{O}_X)$, we have a canonical functor

$$\begin{aligned} \Gamma(X, \bullet) : \mathcal{O}_X\text{-}\mathbf{mod} &\to A\text{-}\mathbf{mod} \\ \mathcal{F} &\mapsto \Gamma(X, \mathcal{F}), \end{aligned}$$

where A is the ring $\Gamma(X, \mathcal{O}_X)$.

Examples of Locally Ringed Spaces

- **Example 1: Affine Scheme**

  Let A be a ring, and $X = \mathrm{Spec}(A)$. Then there exists a sheaf of rings $\tilde{A}$ such that for every $f \in A$, $\tilde{A}(\mathrm{D}(f)) = A_f$, where $\mathrm{D}(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) \,|\, f \notin \mathfrak{p}\}$. Then $(\mathrm{Spec}(A), \tilde{A})$ is a locally ringed space.

  Definition. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(A), \tilde{A})$, for some ring A. In fact $A = \Gamma(X, \mathcal{O}_X)$.

- **Example 2: Topological Space**

  Let X be a topological space, and let $\mathcal{C}_X$ denote the sheaf of continuous real valued functions on X. Then, $(X, \mathcal{C}_X)$ is a locally ringed space.

  Similarly $(X, \mathcal{C}^\infty_X(\mathbf{R})$, $(X, \mathcal{C}^\infty_X(\mathbf{C}))$ are examples of locally ringed spaces.

The Serre-Swan Theorem

- Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let A denote the ring $\Gamma(X, \mathcal{O}_X)$.

- **Definition.** We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if for every $x \in X$, there exist an open neighborhood U of x, and a set I such that $\mathcal{F}|_U \cong \mathcal{O}_X^{(I)}|_U$ as an $\mathcal{O}_X|_U$-module.

- For all $x \in X$, let $\mathrm{rk}_x(\mathcal{F}) = \dim_{k(x)}(\mathcal{F}(x))$, where

$$\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) = \mathcal{F}_x / \mathfrak{m}_{X,x}\mathcal{F}_x.$$

- Let $\mathbf{Lfb}(X)$ denote the full subcategory of $\mathcal{O}_X$-$\mathbf{mod}$ consisting of locally free $\mathcal{O}_X$-modules of bounded rank.

- Let $\mathbf{Fgp}(A)$ be the full subcategory of A-$\mathbf{mod}$ consisting of finitely generated projective A-modules.

The Serre-Swan Theorem (Continued)

- **Serre's Theorem [1955].** Let $(X, \mathcal{O}_X)$ be an affine scheme, and let A denote its coordinate ring $\Gamma(X, \mathcal{O}_X)$. Then the functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$, $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ is an equivalence of categories.
- **Swan's Theorem [1962].** Let X be a paracompact topological space of bounded topological dimension, and let $\mathcal{C}_X$ denote the sheaf of continuous real-valued functions on X. Let $C(X)$ denote the ring $\Gamma(X, \mathcal{C}_X)$. Then, the functor $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(C(X))$ is an equivalence of categories.
- We will say that *the Serre-Swan Theorem holds* for a ringed space $(X, \mathcal{O}_X)$ if $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ is an equivalence of categories.

Some Remarks

- **Definition.** A continuous map $\pi : E \to X$ of one Hausdorff space, E, onto another, X, is called a *K-vector bundle*, where K is $\mathbf{R}$ or $\mathbf{C}$, if the following conditions are satisfied:

  1. $E_p := \pi^{-1}(p)$, for $p \in X$, is a K-vector space (E_p is called the *fiber* over p).
  2. For every $p \in X$ there is a neighborhood U of p and a homeomorphism $h : \pi^{-1}(U) \to U \times K^r$ such that $h(E_p) \subset \{p\} \times K^r$, and h^p, defined by the composition $h^p : E_p \xrightarrow{h} \{p\} \times K^r \xrightarrow{\text{pr}_2} K^r$, is a K-vector space isomorphism, for some integer r (the pair (U, h) is called a *local trivialization*).

- Let X be a connected manifold. Let $\mathbf{Vect}(X)$ denotes the category of K-vector bundles (in the above sense) on X, then $\mathbf{Lfb}(X)$ and $\mathbf{Vect}(X)$ are equivalent categories.

The Serre-Swan Theorem for Ringed Spaces

- **Definition.** An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *generated by global sections* if there is a family of sections $(s_i)_{i\in I}$ in $\Gamma(X,\mathcal{F})$ such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. We say that $\mathcal{F}$ is *finitely generated by global sections* if I is finite.

- **Definition.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. Then, a subcategory $\mathcal{C}$ of $\mathcal{O}_X$-**mod** is called an *admissible subcategory* if it satisfies the following conditions:

C1. $\mathcal{C}$ is a full abelian subcategory of $\mathcal{O}_X$-**mod**, and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ belongs to $\mathcal{C}$ for every pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, where $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ denotes the sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$.

C2. Every sheaf in $\mathcal{C}$ is acyclic, and generated by global sections.

C3. $\mathbf{Lfb}(X)$ is a subcategory of $\mathcal{C}$.

The Serre-Swan Theorem for Ringed Spaces (Continued)

- **Theorem.** Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $A = \Gamma(X, \mathcal{O}_X)$. Assume that $\mathcal{O}_X$-**mod** contains an admissible subcategory $\mathcal{C}$, and that every sheaf in $\mathbf{Lfb}(X)$ is finitely generated by global sections. Then, $\Gamma(X, \bullet) : \mathbf{Lfb}(X) \to \mathbf{Fgp}(A)$ is an equivalence of categories, i.e., the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$.

Some Special Cases

- **Serre's Theorem.**

  Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* if for every $x \in X$, there exist an open neighborhood U of x such that there exists an exact sequence of $\mathcal{O}_X|_U$-modules,

  $$\mathcal{O}_X^{(I)}|_U \to \mathcal{O}_X^{(J)}|_U \to \mathcal{F}|_U \to 0,$$

  where I and J are arbitrary index sets.

  Let $(X, \mathcal{O}_X)$ be an affine scheme, with a coordinate ring A. Let $\mathbf{Qcoh}(X)$ denote the full subcategory of $\mathcal{O}_X$-$\mathbf{mod}$ consisting of quasicoherent $\mathcal{O}_X$-modules. Then $\mathbf{Qcoh}(X)$ is an admissible subcategory of $\mathcal{O}_X$-$\mathbf{mod}$. Since X is quasicompact, locally free sheaves of finite rank are finitely generated by global sections. Therefore the Serre's Theorem will follows from the main Theorem.

Some Special Cases (Continued)

- **Swan's Theorem.**
  On a paracompact space fine sheaves are acyclic, and generated by global sections. Since $\mathcal{C}_X$ is a fine sheaf, $\mathcal{C}_X$-**mod** is an admissible category. Since vector bundles are finitely generated by global sections on a paracompact space of bounded topological dimension is a standard result, the Swan's Theorem follows from the main theorem.

- **Remark.** Let $(X, \mathcal{O}_X)$ be a ringed space such that, X is a paracompact topological space of bounded topological dimension, and $\mathcal{O}_X$ is a fine sheaf. Then, the Serre-Swan Theorem holds for $(X, \mathcal{O}_X)$.

Some Special Cases (Continued)

- **Stein Spaces [Forster 1967].**

  Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if it satisfies the following conditions:

  1. $\mathcal{F}$ is of finite type.
  2. For every open subsets U of X, for every integer p, and for every morphism of $\mathcal{O}_X|_U$-modules $u : \mathcal{O}_X^p|_U \to \mathcal{F}|_U$, the $\mathcal{O}_X|_U$-module $\ker(u)$ is of finite type.

  Recall that a complex space $(X, \mathcal{O}_X)$ is called a *Stein space* if every coherent sheaf is acyclic, (**Theorem B** is valid for $(X, \mathcal{O}_X)$). And we have **Theorem A** for Stein spaces which tells that every coherent sheaf is generated by global sections. Let $(X, \mathcal{O}_X)$ be a finite-dimensional connected Stein space. Let $\mathbf{Coh}(X)$ be the category of coherent sheaves over X.

Some Special Cases (Continued)

- **Stein Spaces [Forster 1967]. (Continued)**
  Hence for a Stein spaces $\mathbf{Coh}(X)$ is canonically an admissible subcategory of an $\mathcal{O}_X$-$\mathbf{mod}$. Further, if X is finite dimensional and connected the locally free sheaf of bounded rank are finitely generated by global sections. Hence by the main Theorem we get that the Serre-Swan Theorem holds for a finite-dimensional connected Stein space.

  In particular, if X is a connected noncompact Riemann surface then the Serre-Swan Theorem hold for X. But on the contrary for compact Riemann surfaces the result is not true.

More Examples

- **Affine differentiable spaces**

  Any closed ideal $\mathfrak{a}$ of the Fréchet algebra $\mathcal{C}^\infty(\mathbf{R})$ defines a differentiable space (X, A), where $X = \{x \in \mathbf{R}^n \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}$ is the underlying topological space, and $A = \mathcal{C}^\infty(\mathbf{R}^n)/\mathfrak{a}$ is the algebra of differentiable functions on this differentiable space (X, A) is a basic example of a differentiable space of $\mathbf{R}^n$.

  Lie group G acting on a smooth manifold M, then M/G is a differentiable space. In particular orbifolds are differentiable spaces. (May not be an affine differentiable spaces).

- **Compact locally ringed spaces**

  A ringed space $(X, \mathcal{O}_X)$ is said to be *compact* provided that, the topological space X is compact, and that for every $x, x' \in X$, there exists an element $a \in \Gamma(X, \mathcal{O}_X)$ satisfying $a(x) = 1$ and $a(x') = 0$

- **Regular ringed spaces**

  A ringed space $(X, \mathcal{O}_X)$ is called *regular ringed space* if X is a profinite space, i.e., a compact totally disconnected space, and $\mathcal{O}_{X,x}$ is a field for every $x \in X$.