Dynamics of Distal Group Actions

Riddhi Shah (SPS, JNU)

"Indian Women and Mathematics"

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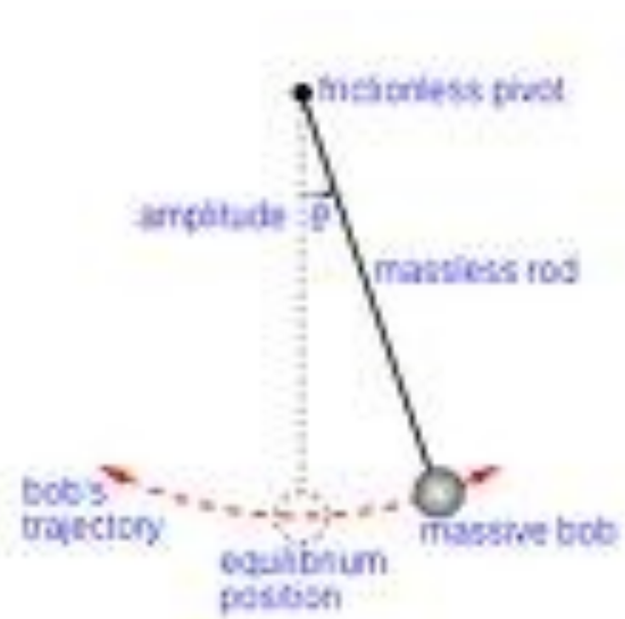
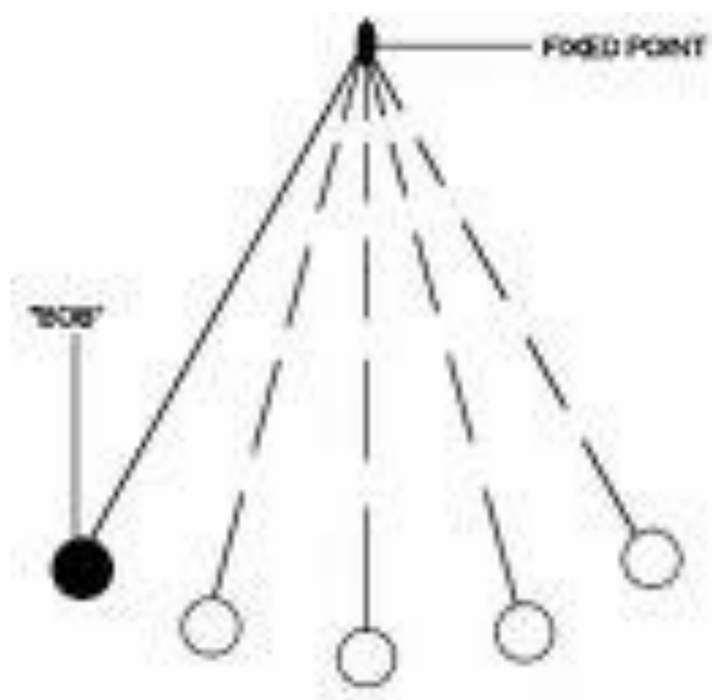
It could be regular or chaotic. We discuss some specific examples now.

Single Pendulum

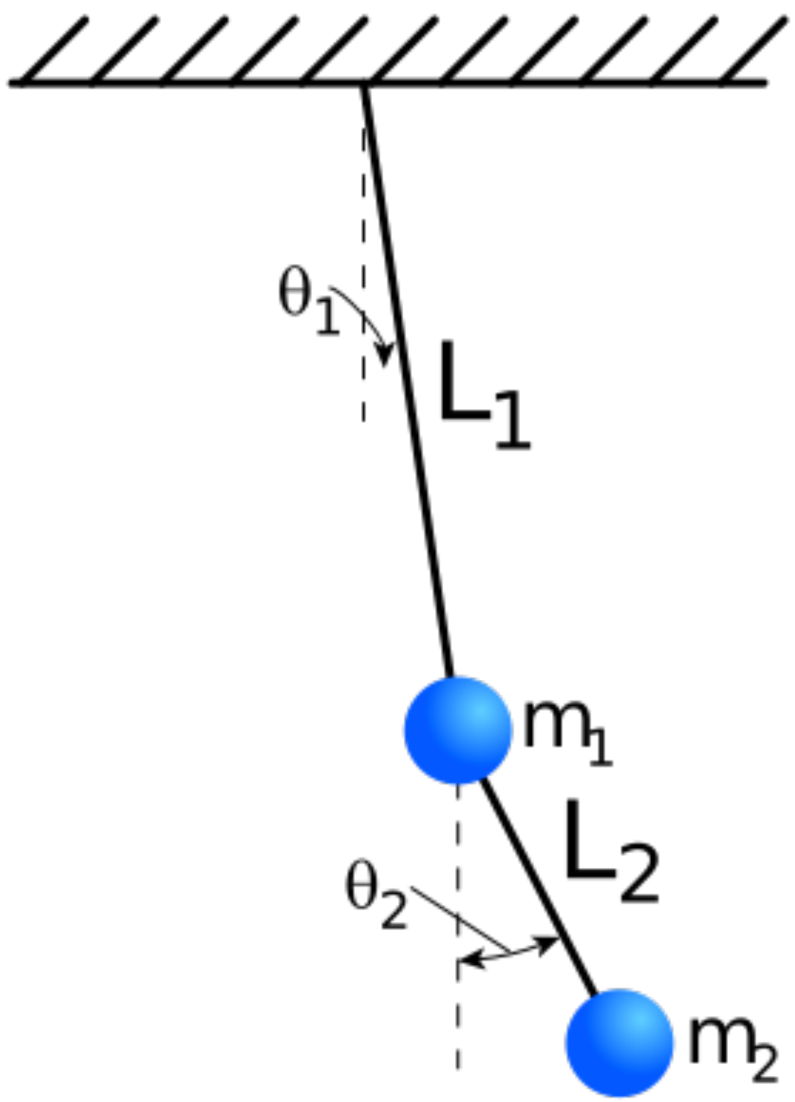
- Single Pendulum behaves nicely.





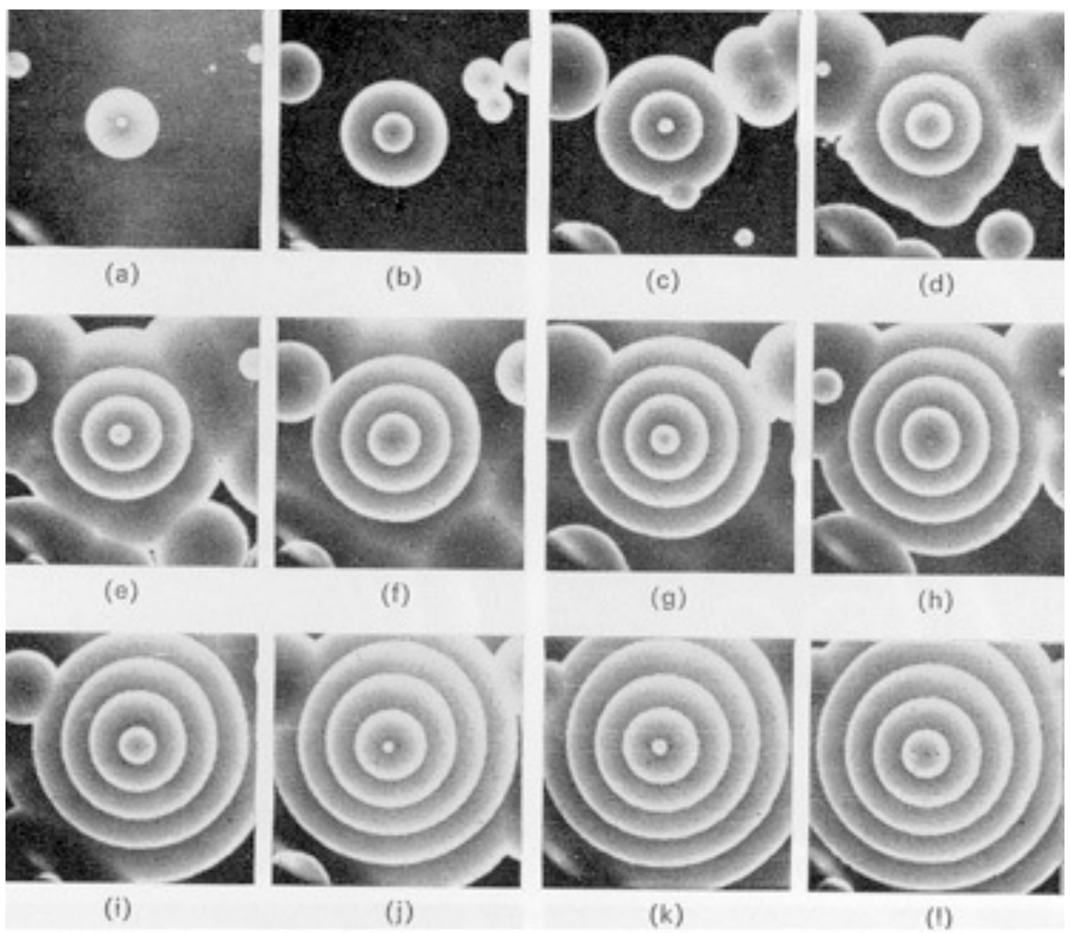
Double Pendulum

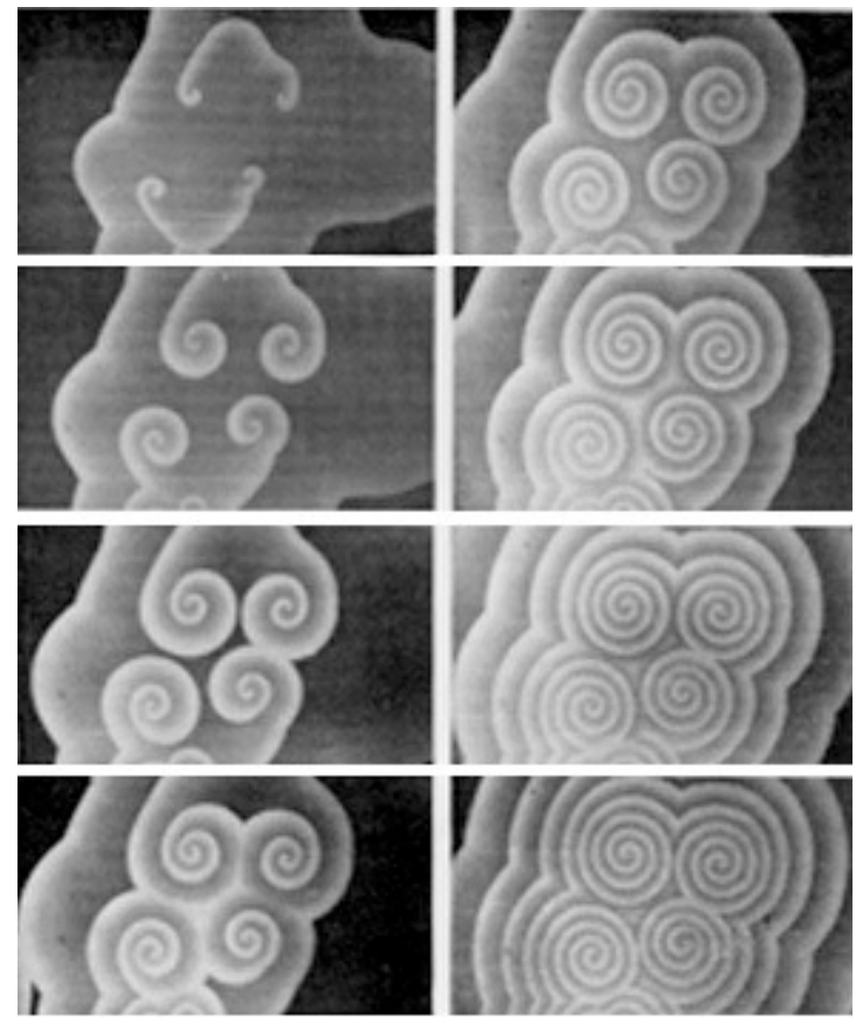
- But double pendulum displays chaos.



Belousov-Zhabotinsky reaction

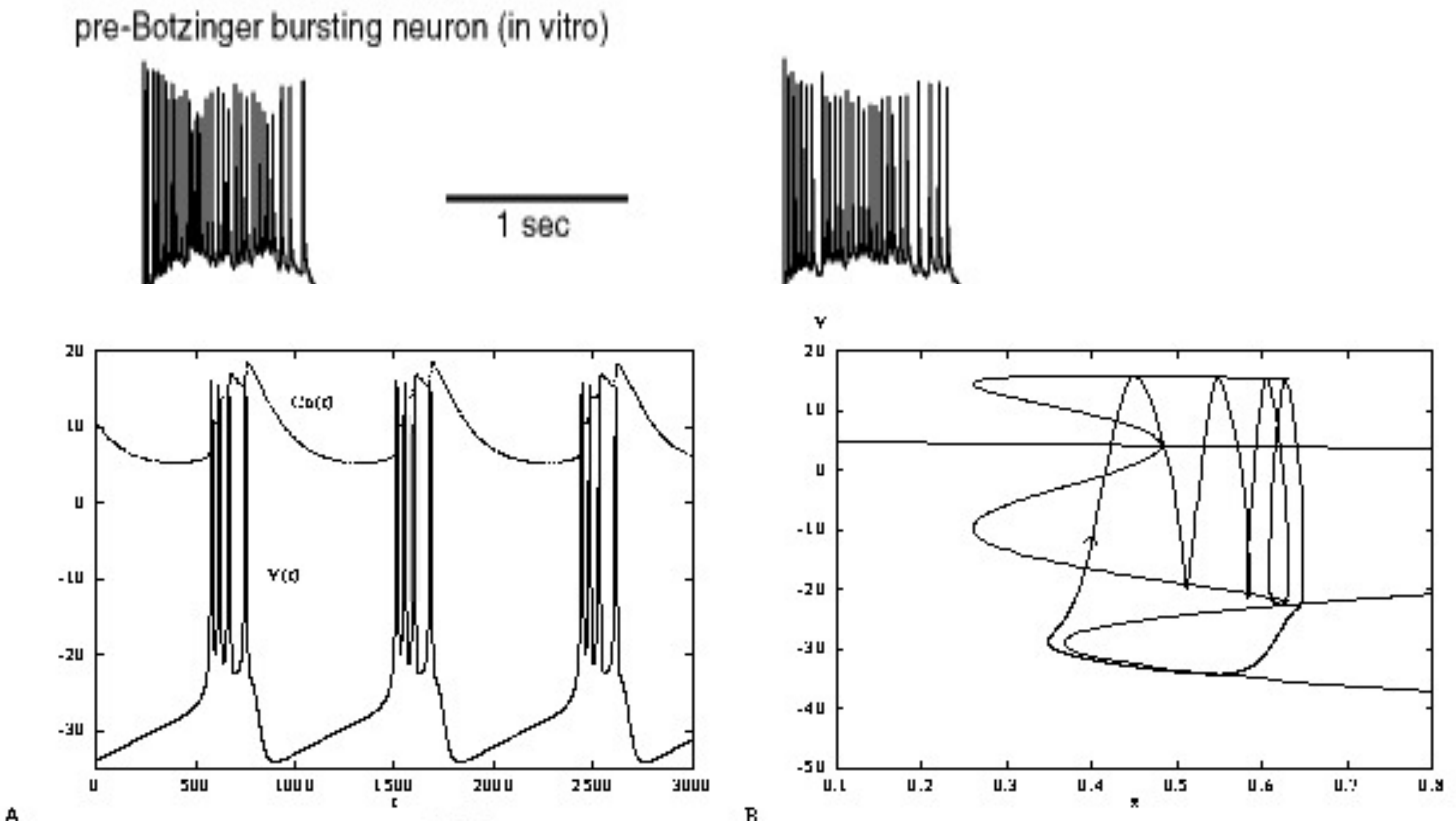
- Describes changes in chemical concentrations (Ce^{3+}/Ce^{4+} couple as catalyst and Citric Acid as reactant)
- Model using reaction diffusion equations $u_t = \nabla^2 u + f(u)$.





Mathematical models for neuronal

- Goal is to model how the voltage across a cell membrane changes over time (My former colleague Amitabh Bose works in this field).
- Hodgkin and Huxley derived equations in 1950’s to describe this. They found that neurons behave almost like electrical circuits.
- Example trace from a bursting neuron related to breathing.



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- In case the map is ergodic (chaotic), there is one orbit of T which is dense in X (like in the path of double pendulum) i.e. $\{T^n(x):n=1,2,3\ldots\}$ is dense or $\{T_t(x)\}$ is dense in X. It is not very sensitive to initial conditions.

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- Distal maps were introduced by Hilbert and studied by many mathematicians.

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- It facilitates the study of nonergodic maps.
- Also the space of probability measures on distal groups (conjugacy maps are distal) have special properties, for. e.g. on compact groups, discrete groups, abelian groups (real line), finite groups etc.
- We will give a splitting of the space into invariant ergodic components as we shall see later.

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- Here, $T^n(x) = x$ or $-x$ for all \$n\$.
- If $|r|<1$, then T contracts any point and orbit of any point goes to zero.
- Namely, $T^n(x) = r^n x \to 0$ as n tends to infinity all \$x\$. So it can not be distal.

Rotations on Unit Circle S^1

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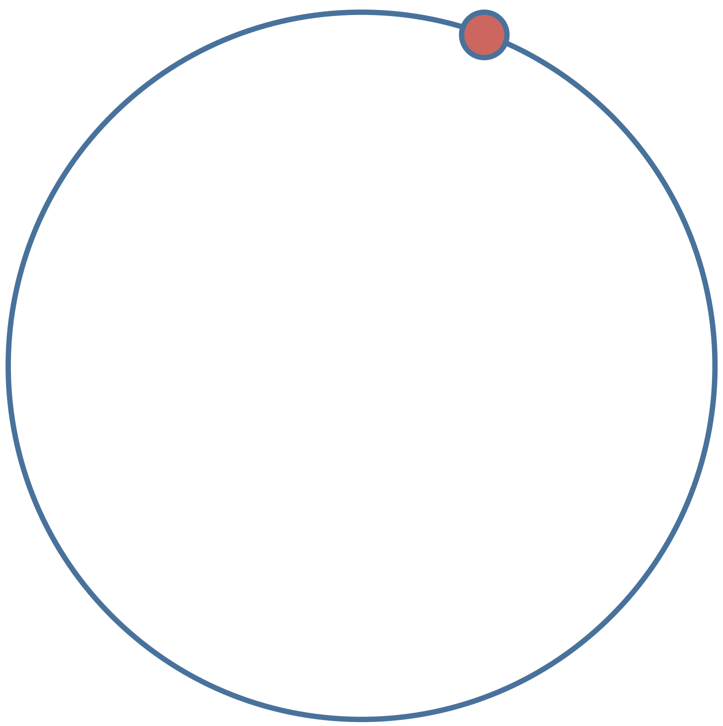
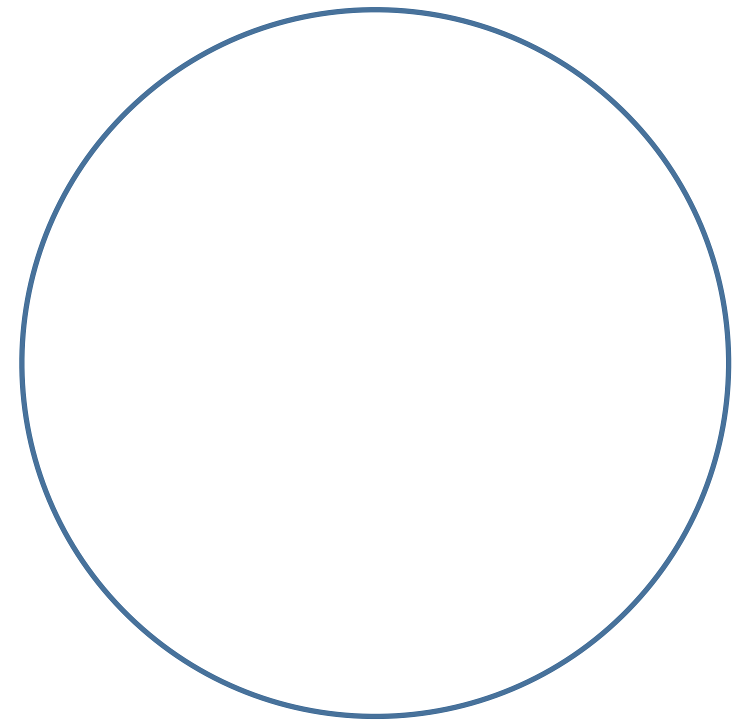
- In most cases, ergodicity and distality are two opposite phenomena.
- We next have a pictorial description of rotation maps thanks to my former colleague Amitabh Bose .

Periodic orbit



Dense orbit



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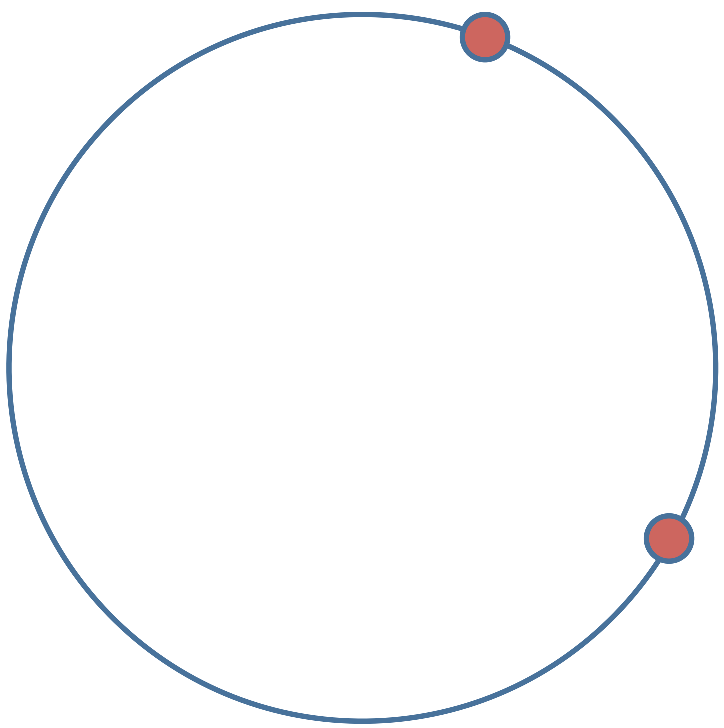

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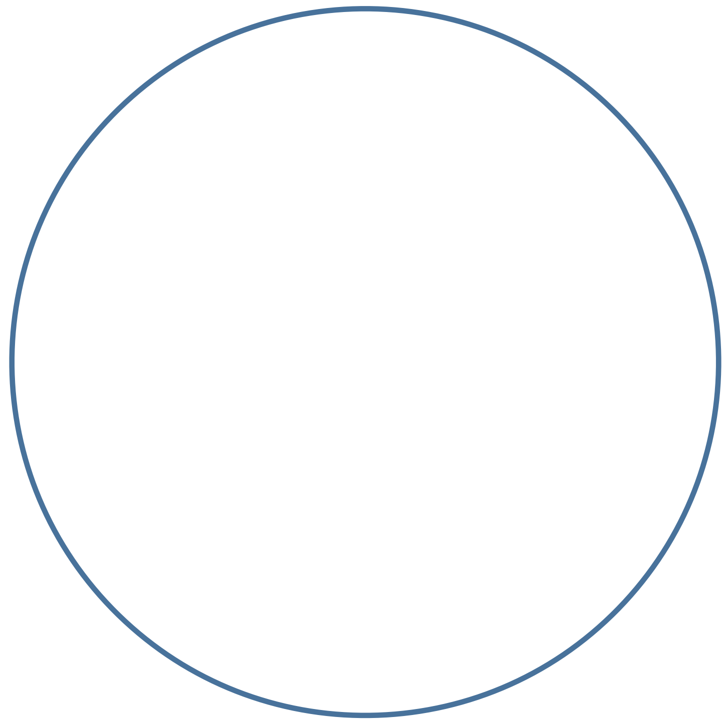

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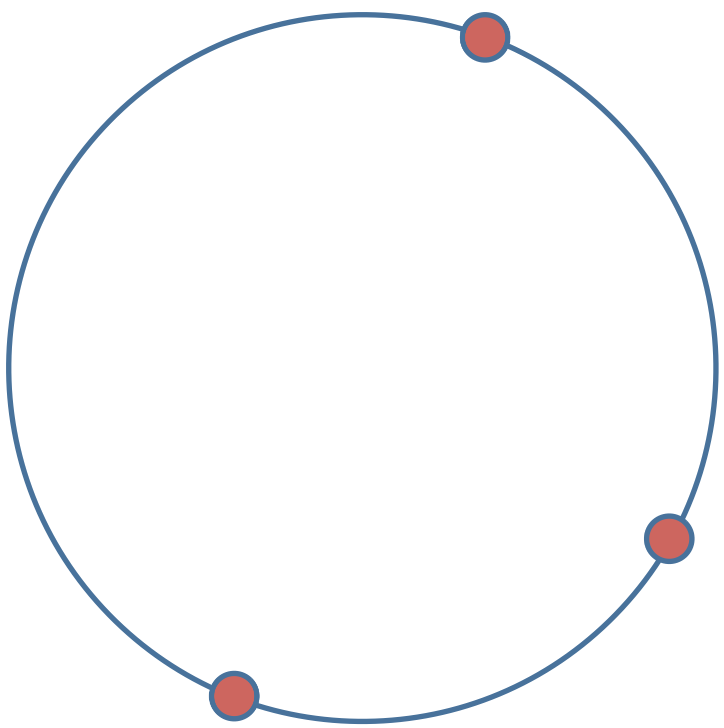

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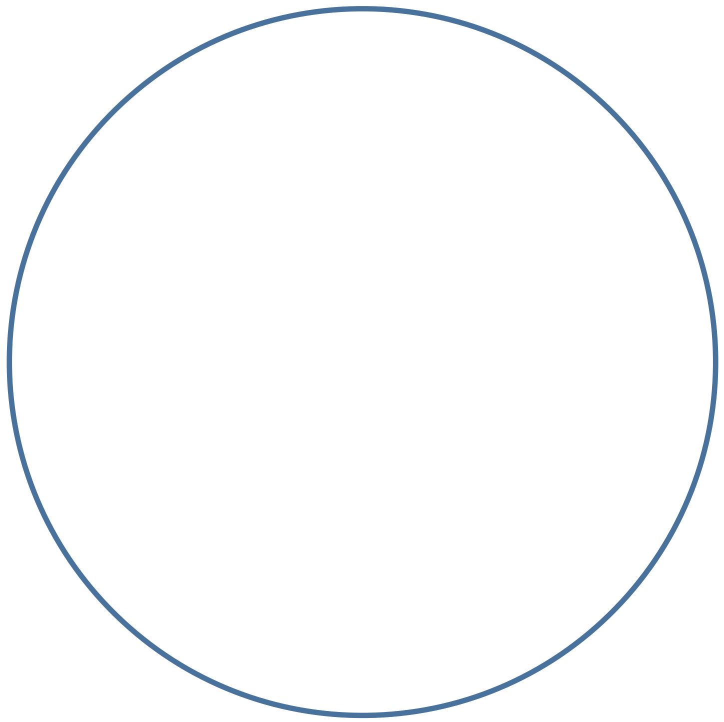

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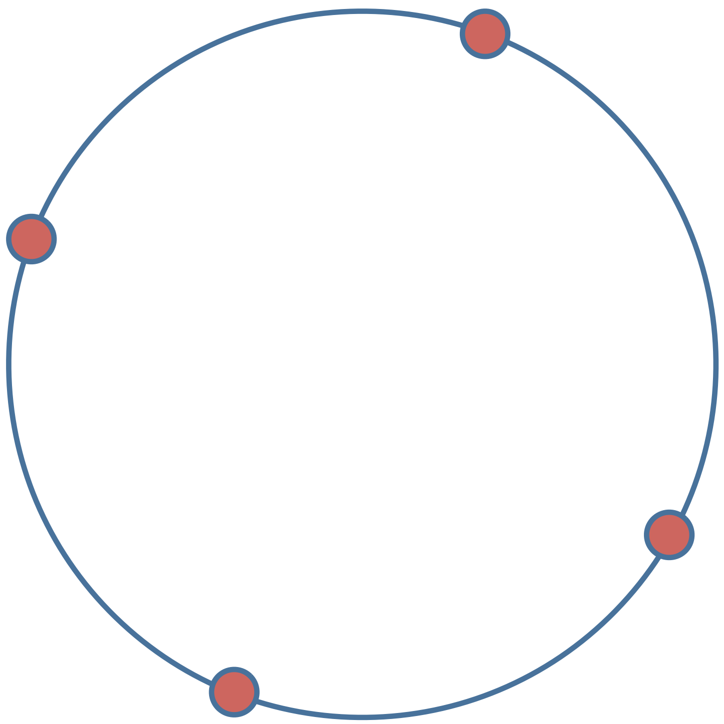

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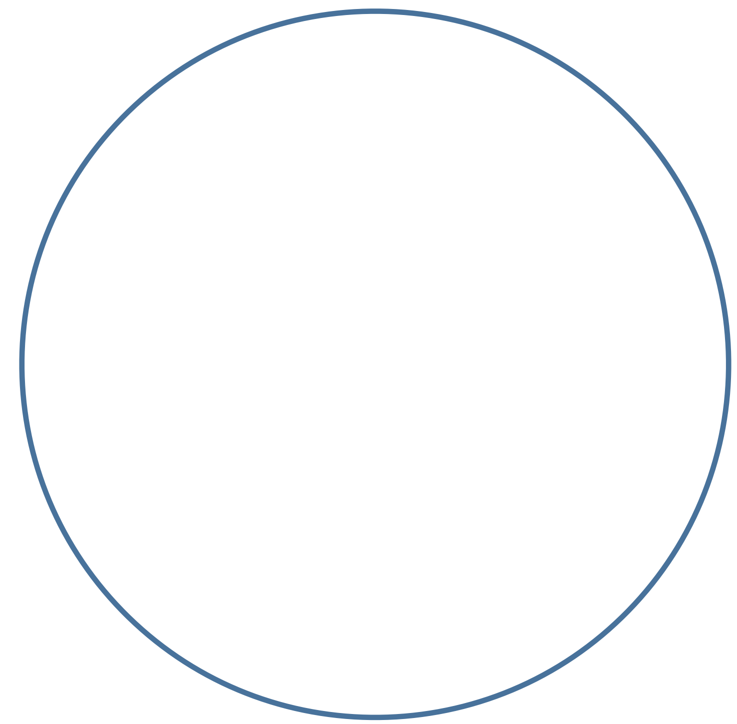

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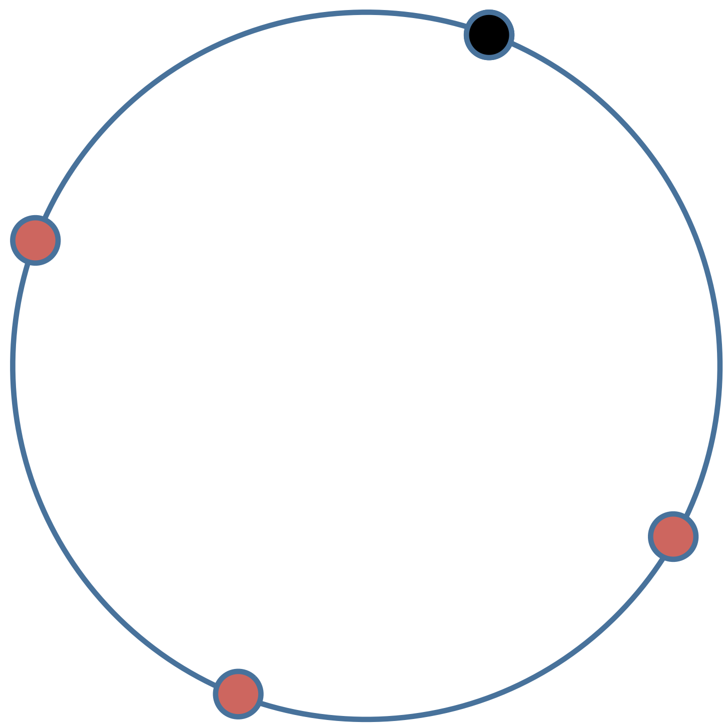

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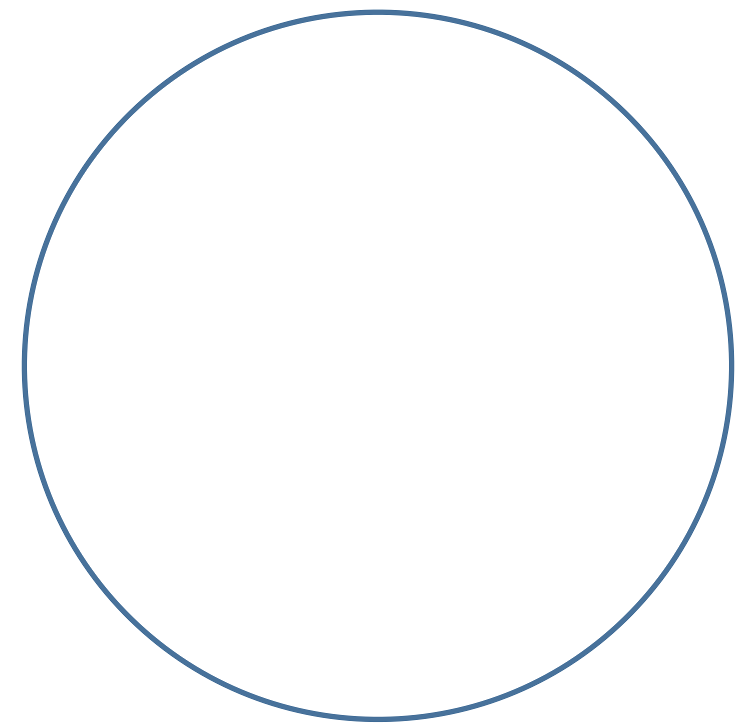

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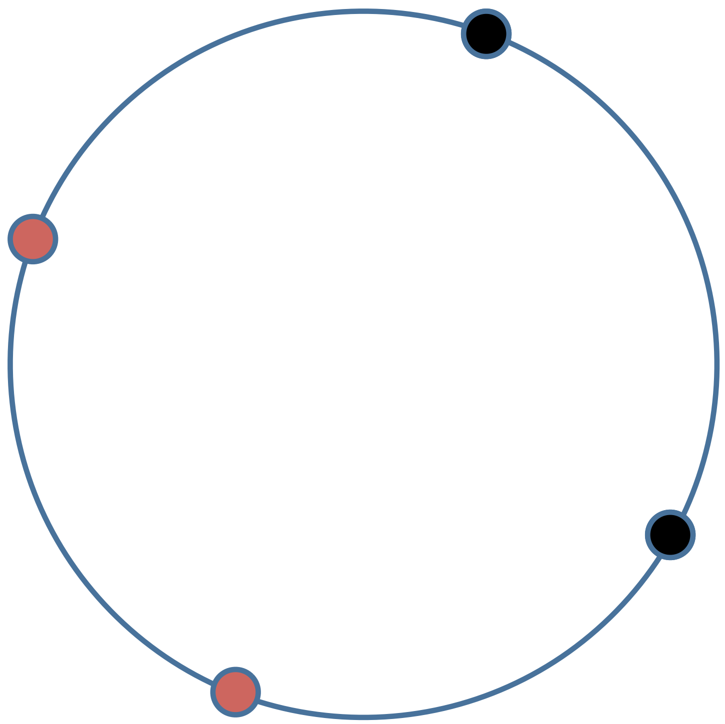

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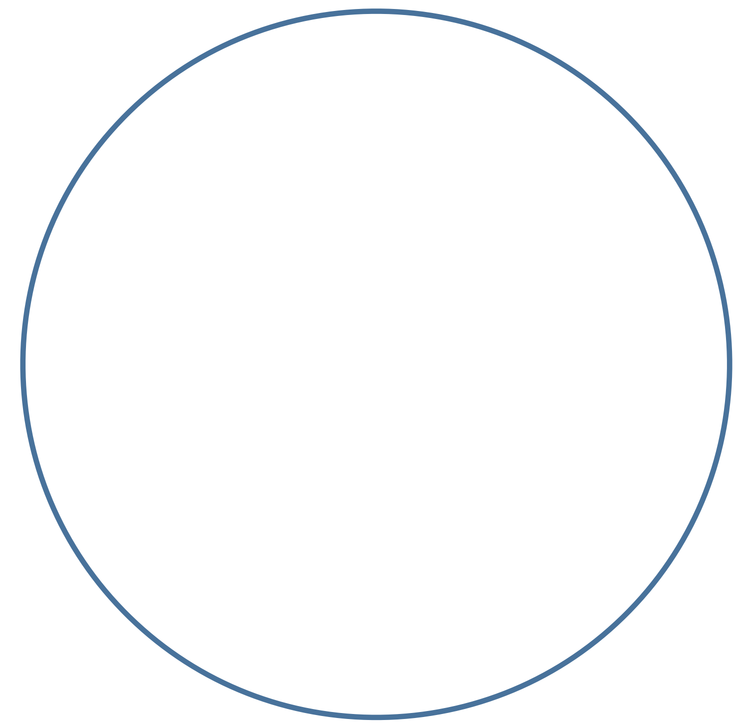

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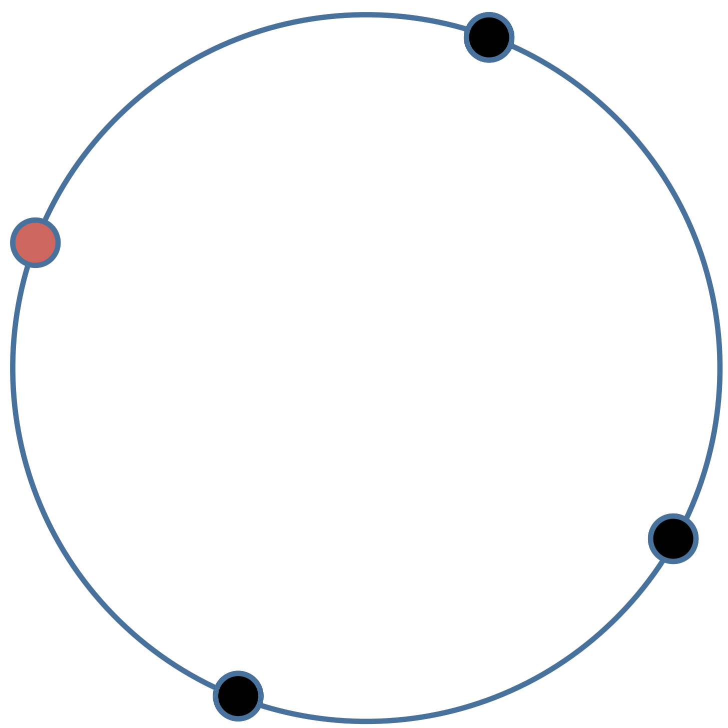

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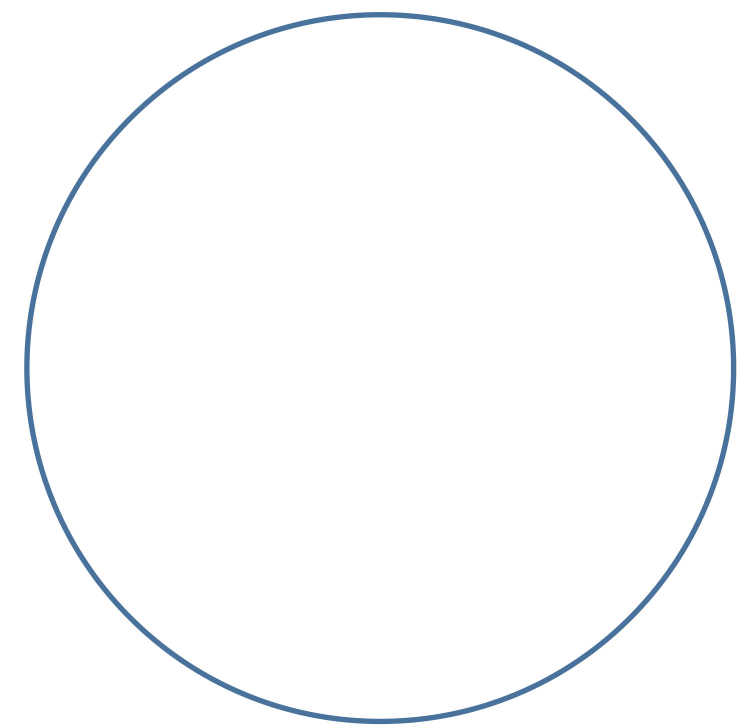

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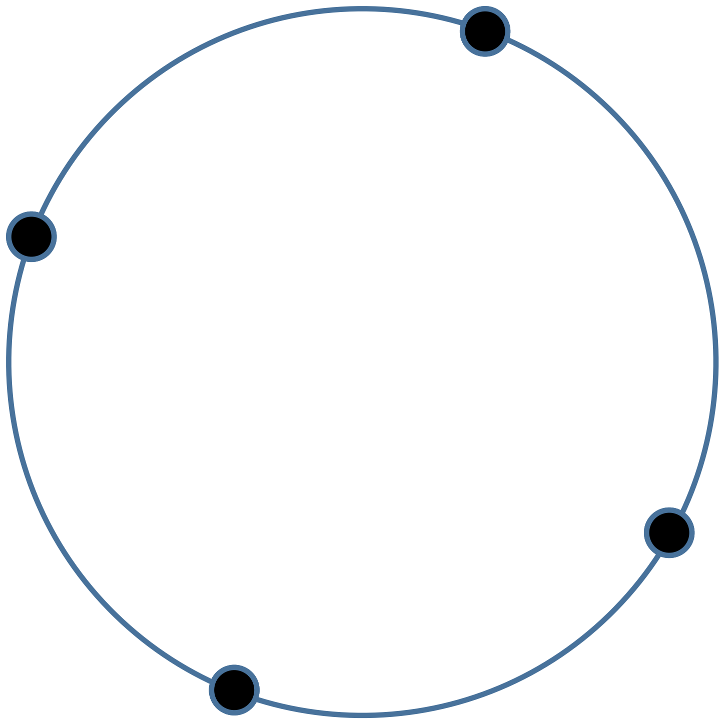

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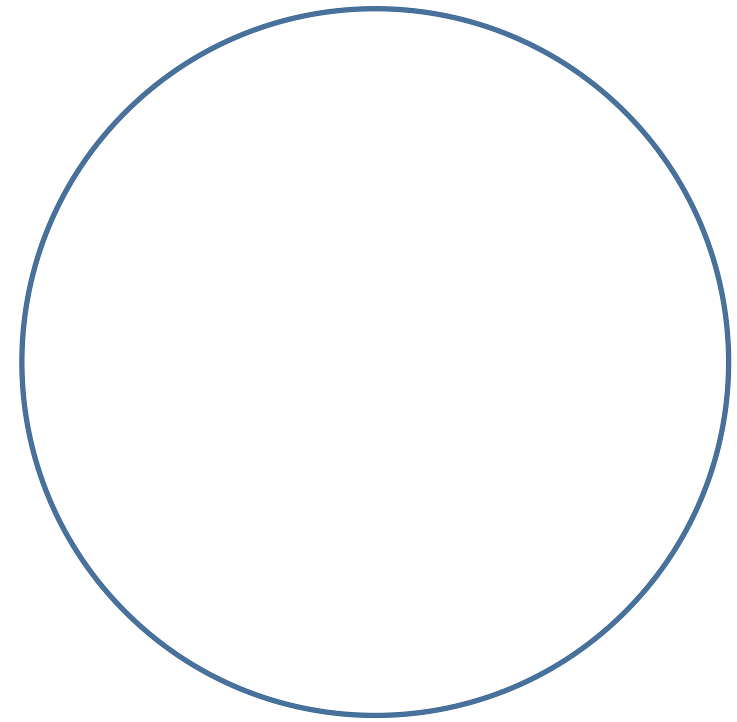

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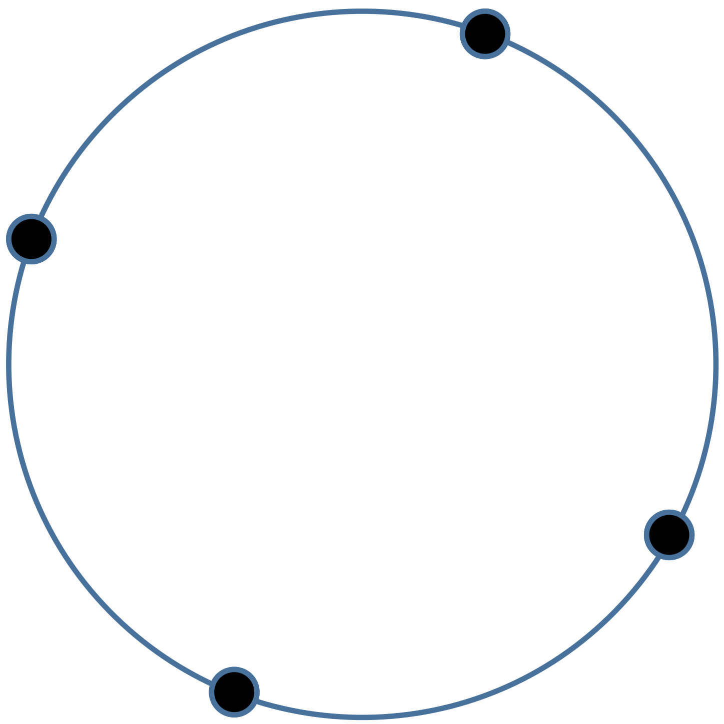

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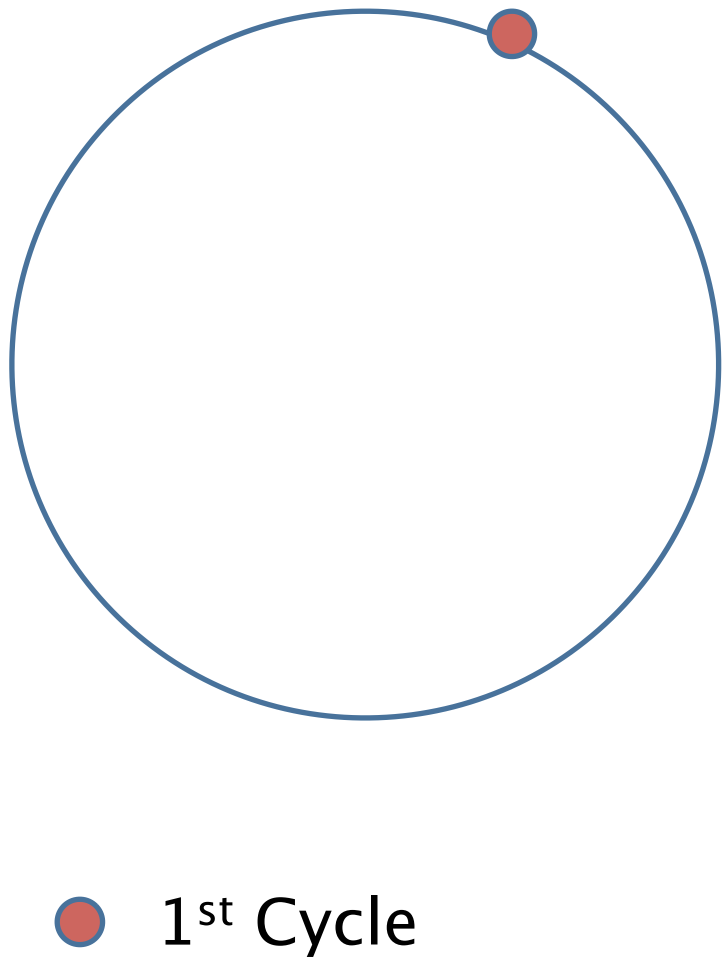

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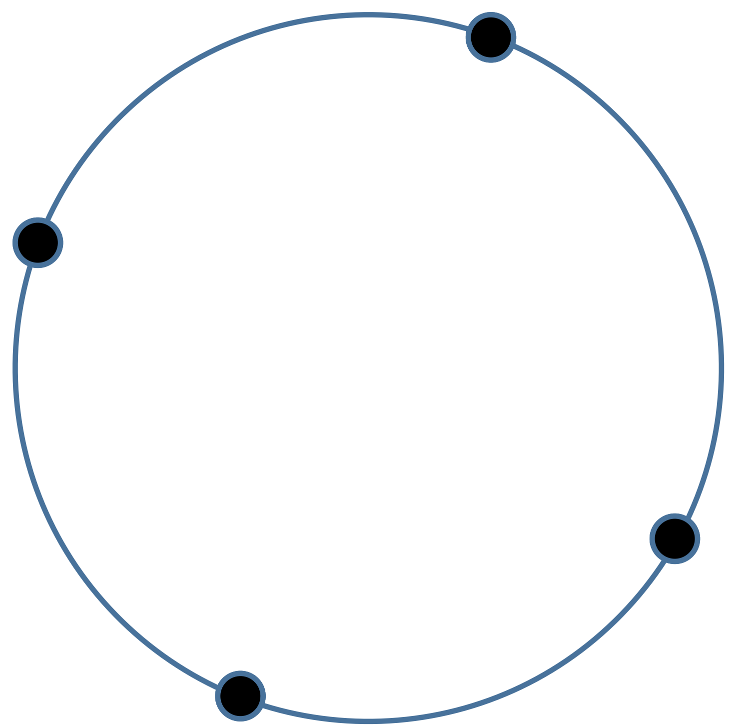

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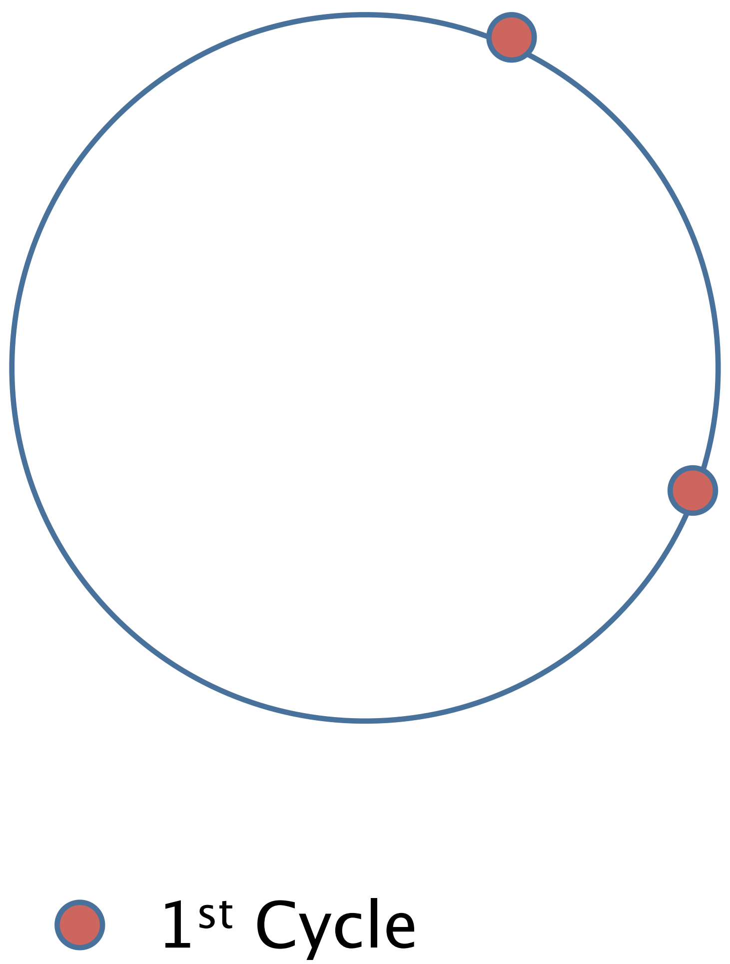

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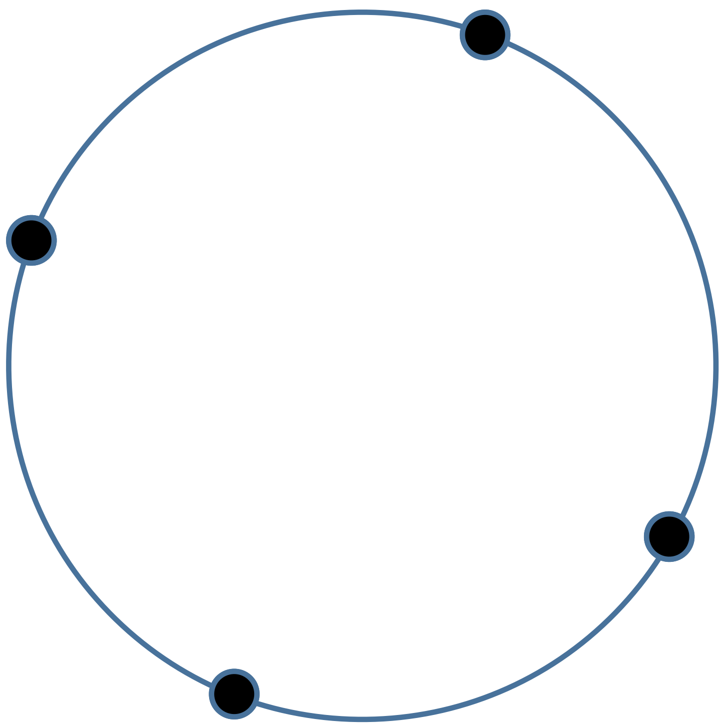

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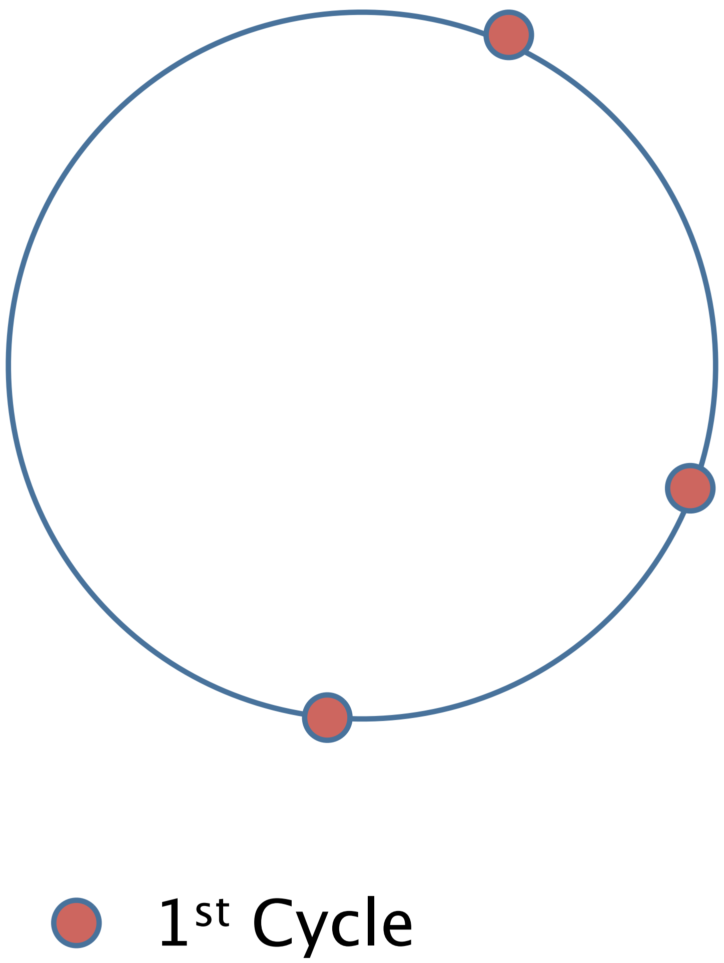

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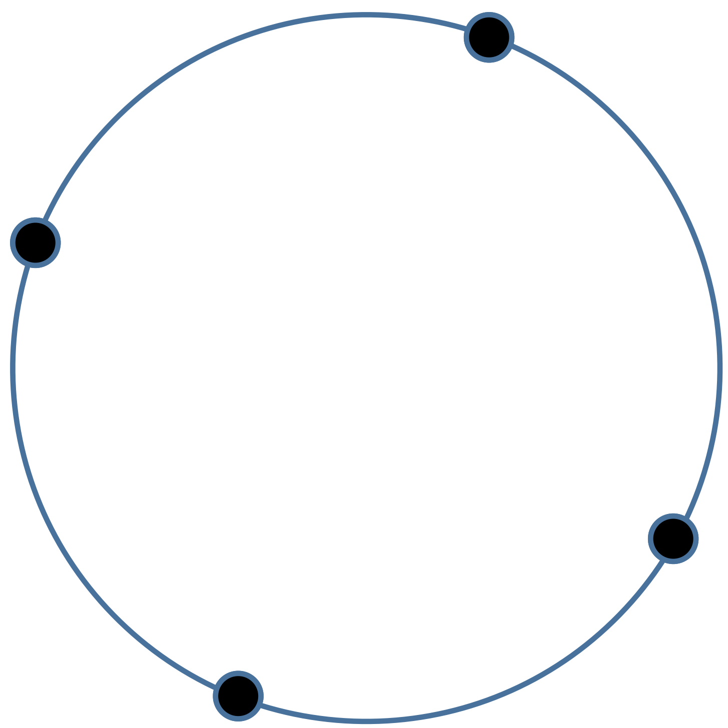

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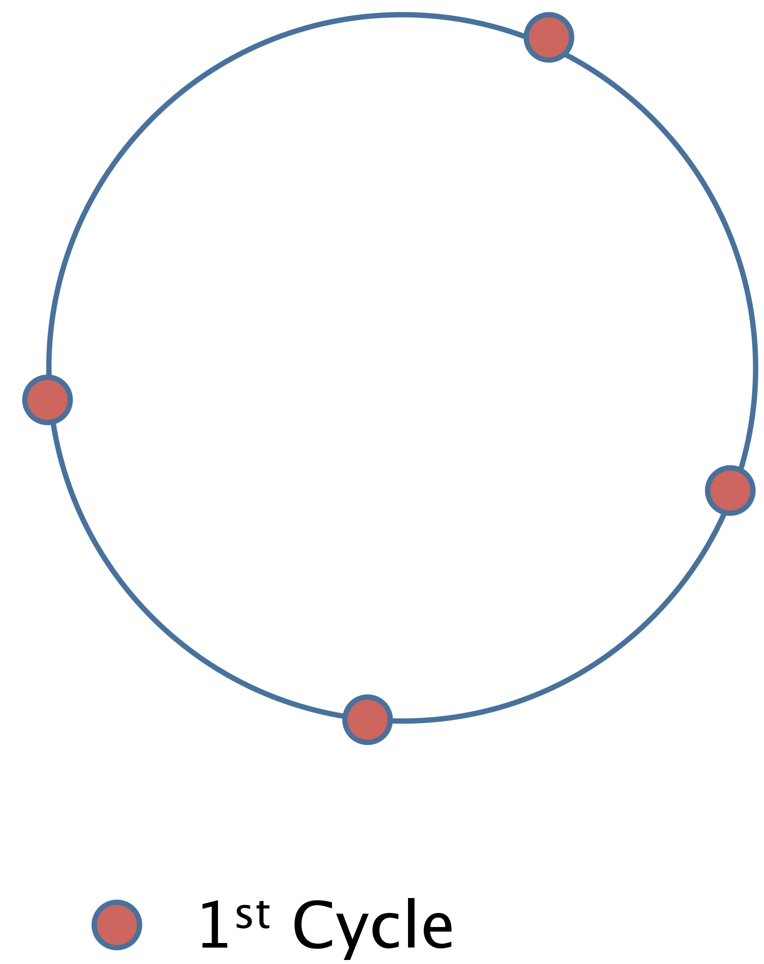

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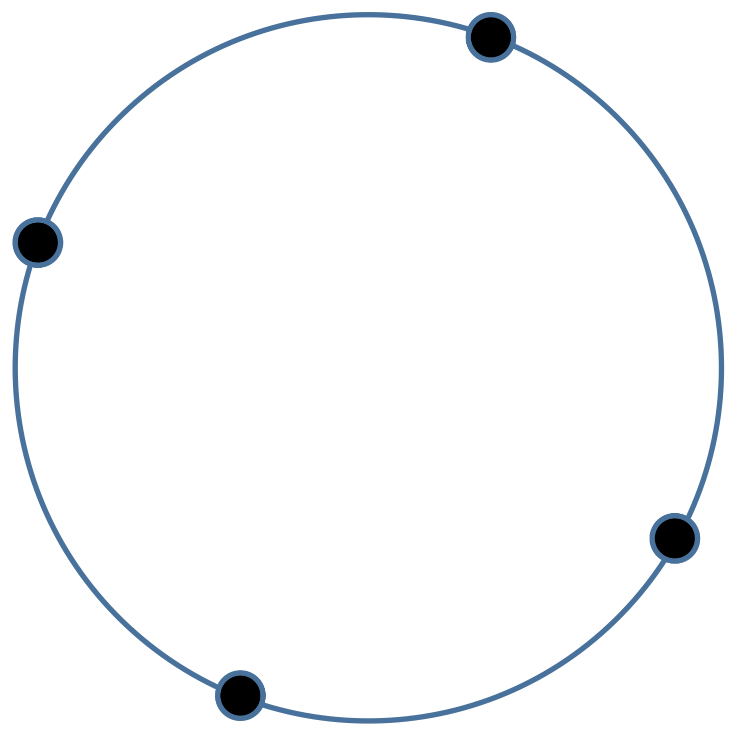

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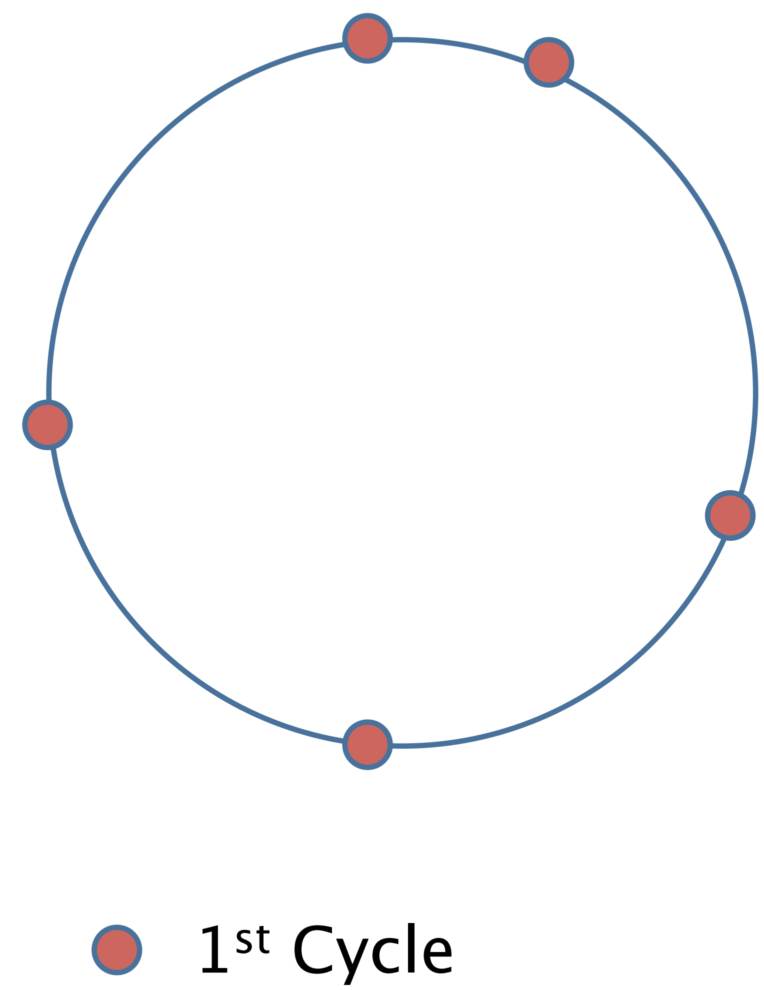

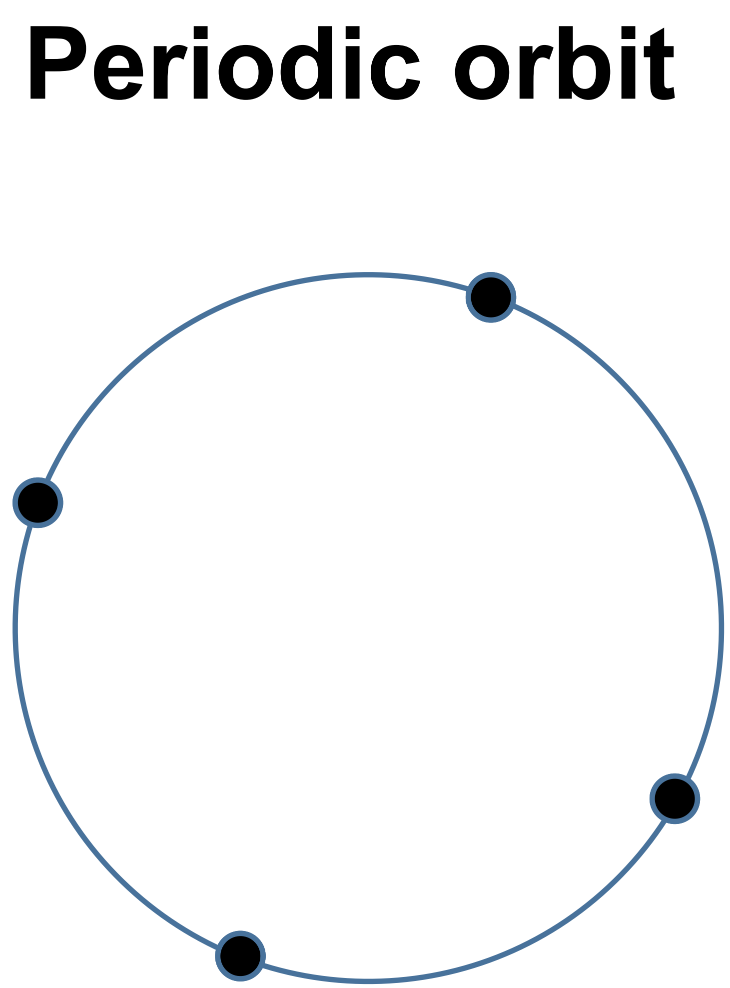
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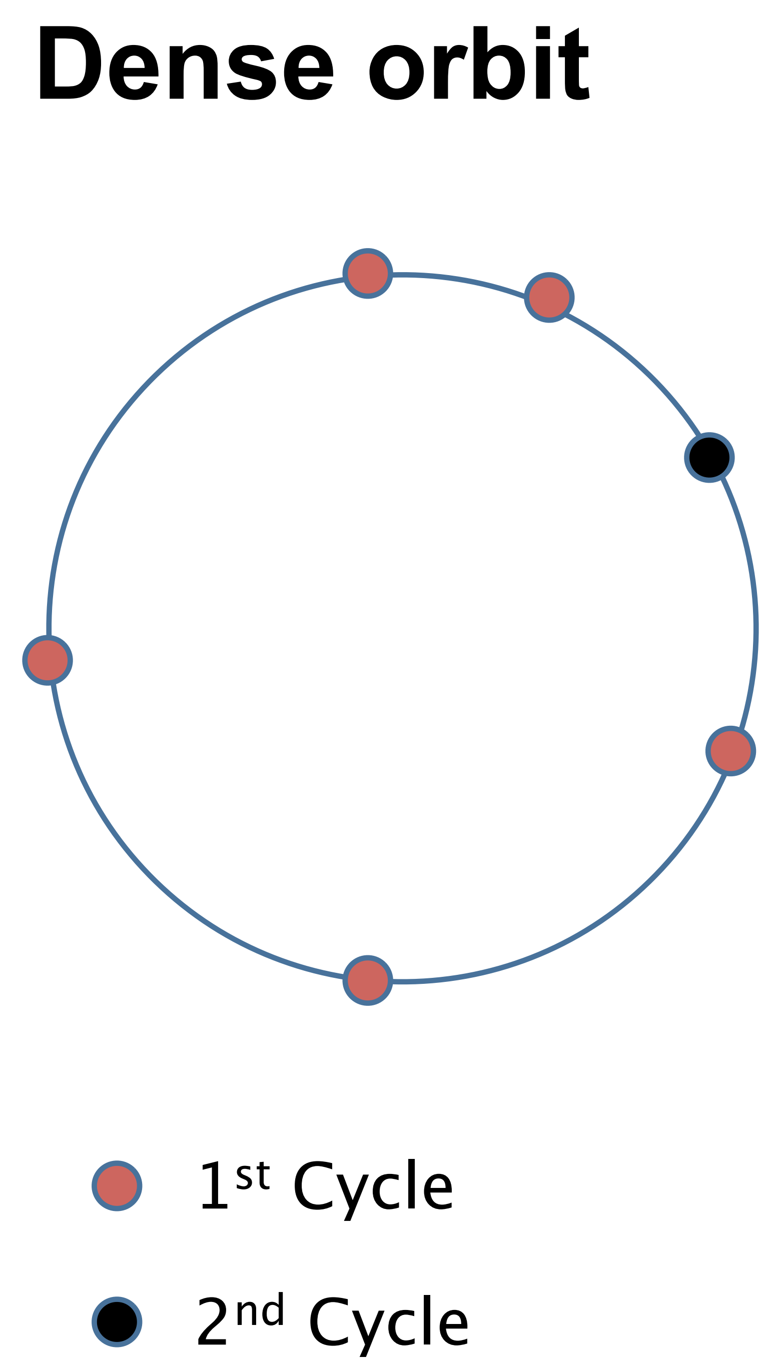
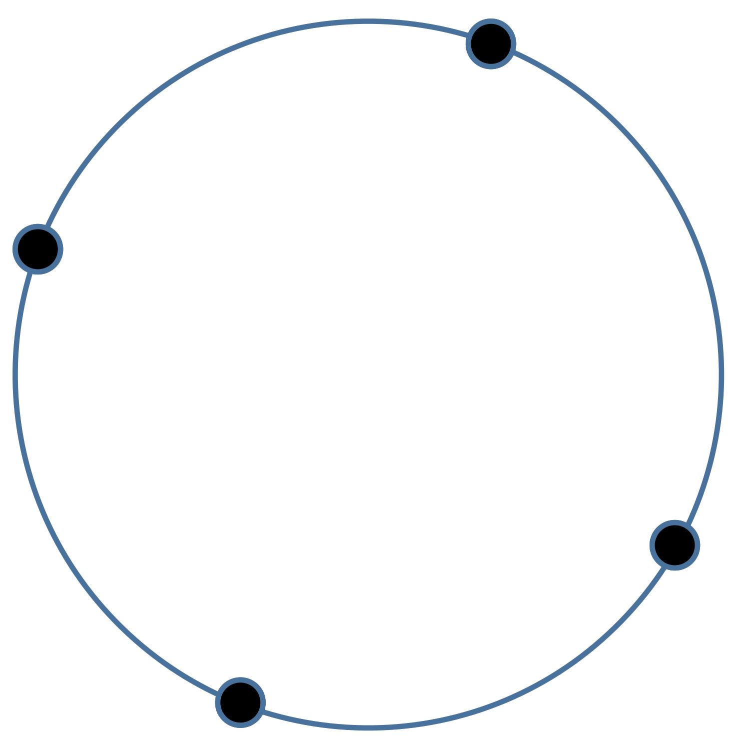

Dense orbit
1st Cycle
2nd Cycle

Periodic orbit



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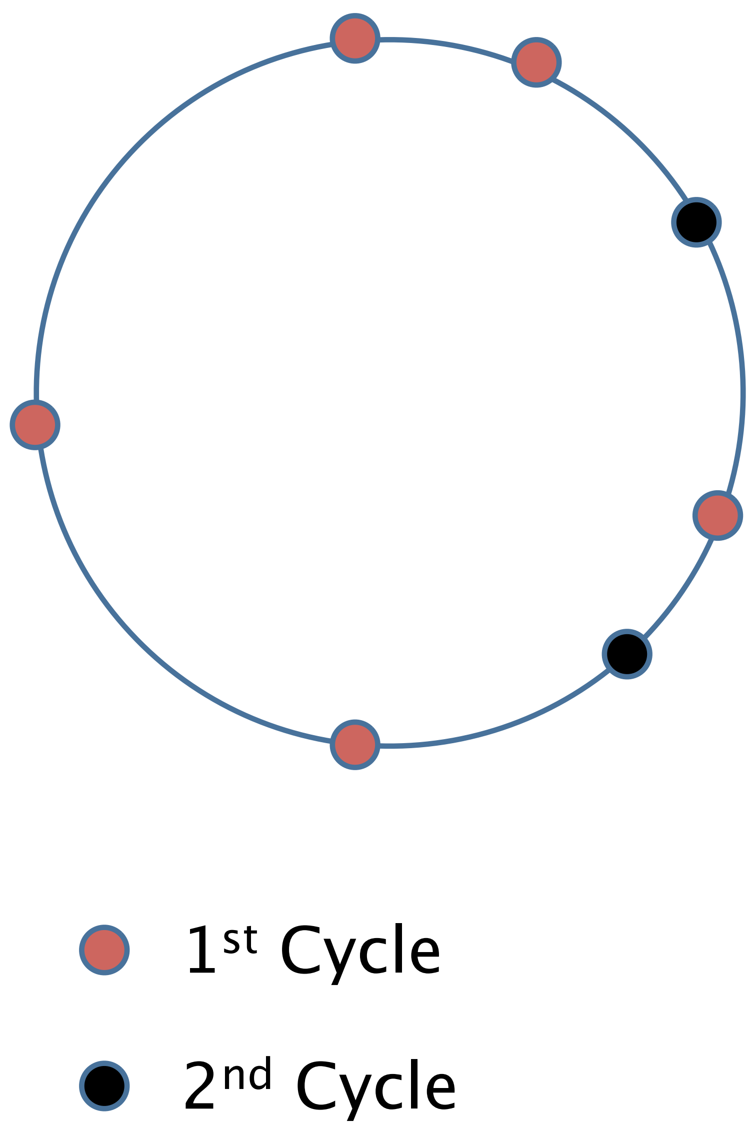

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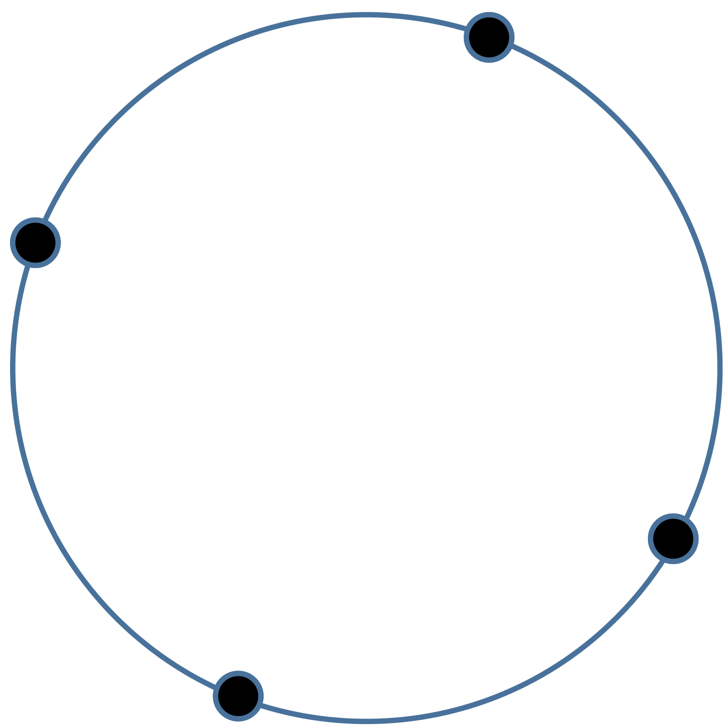

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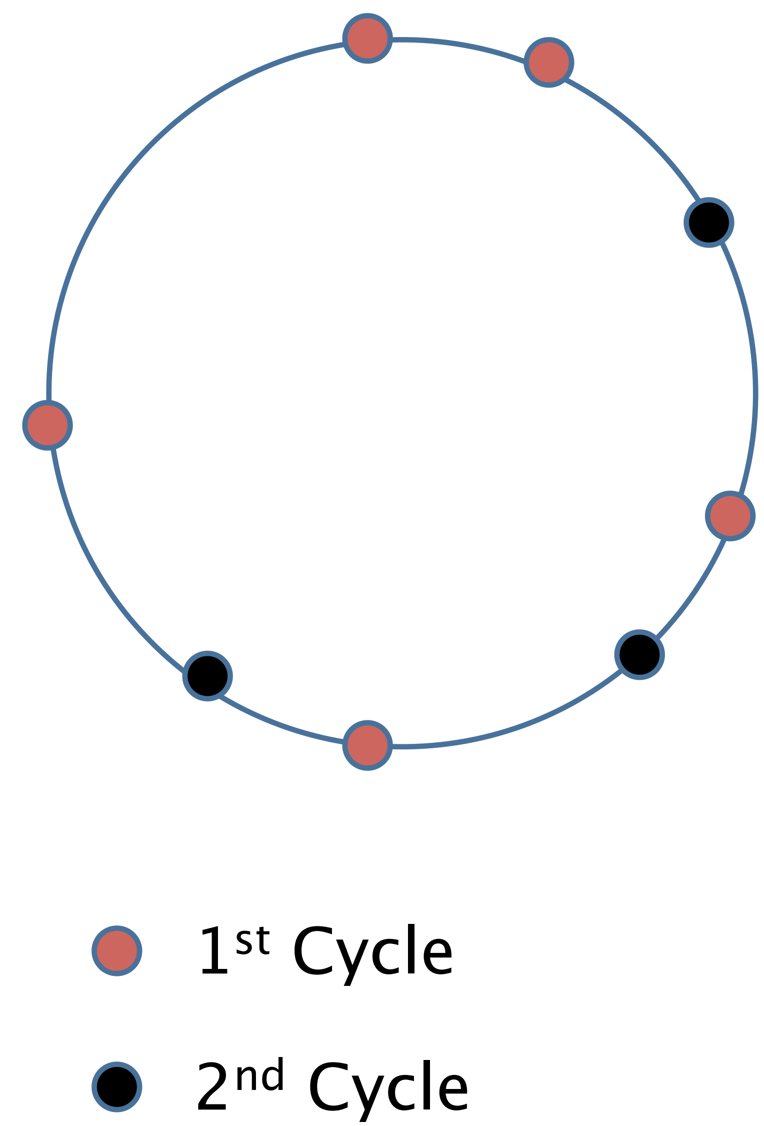

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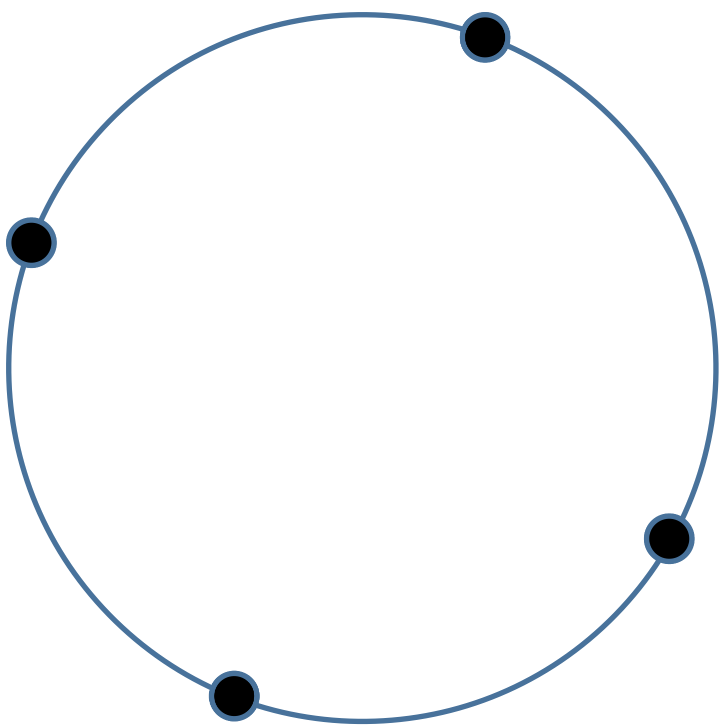

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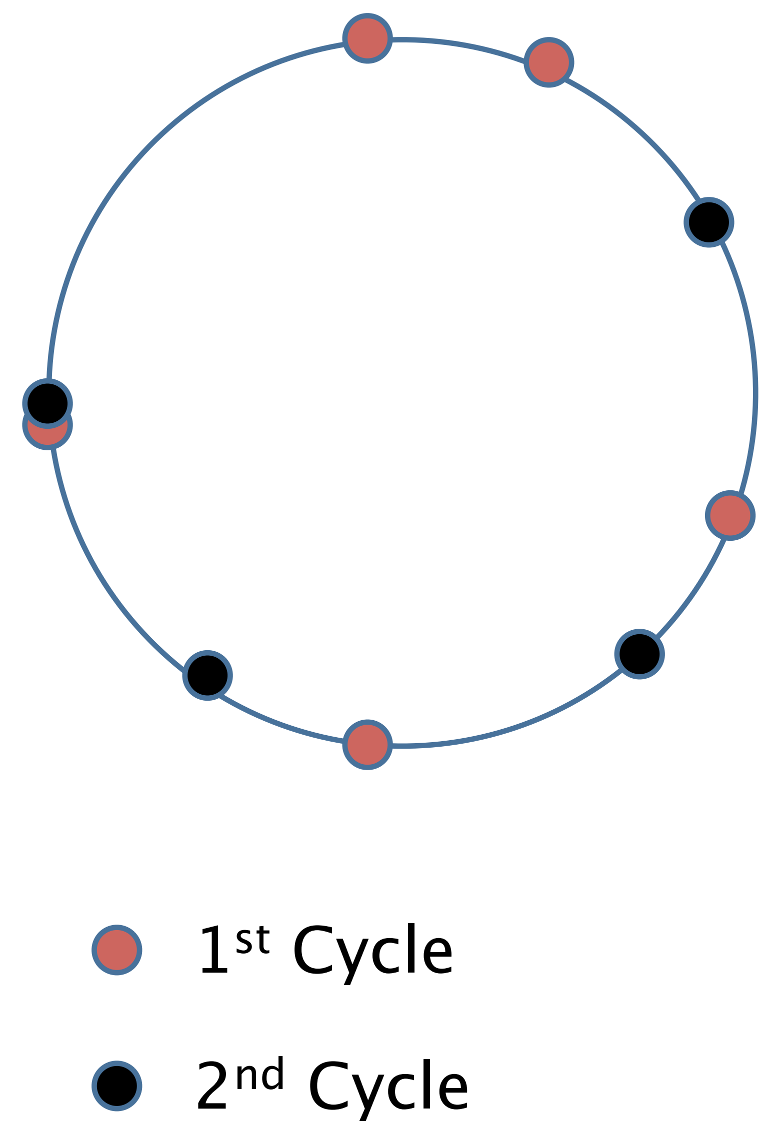

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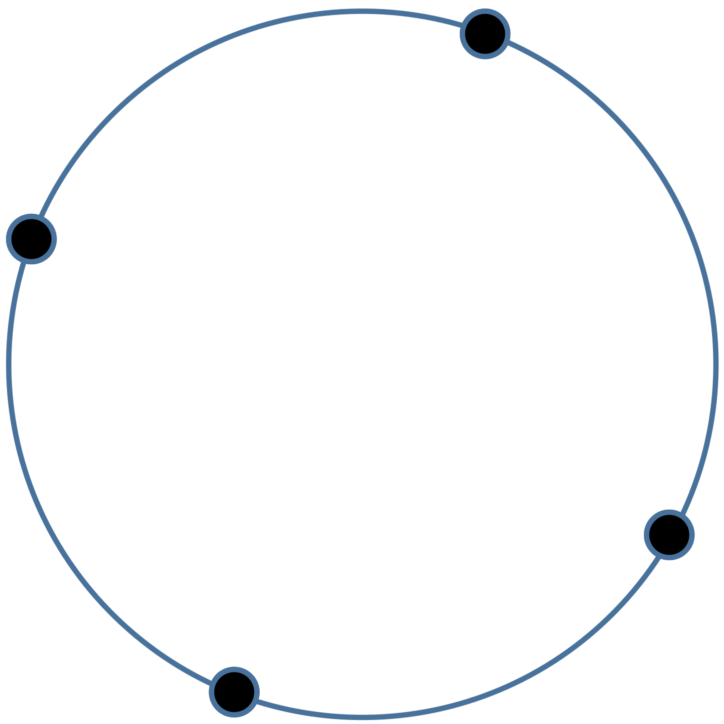

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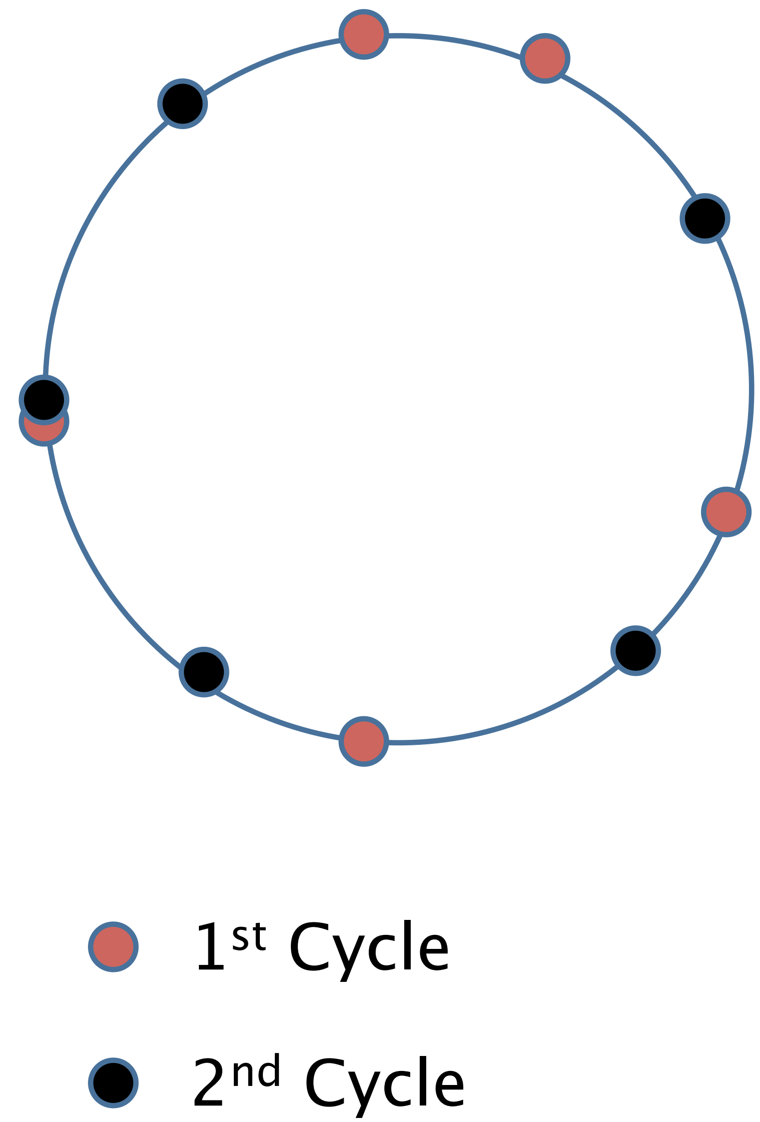

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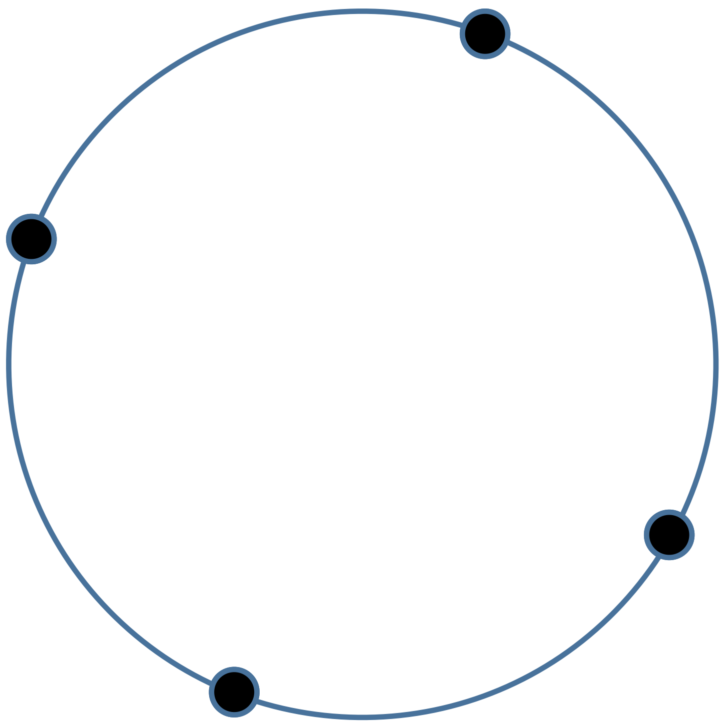

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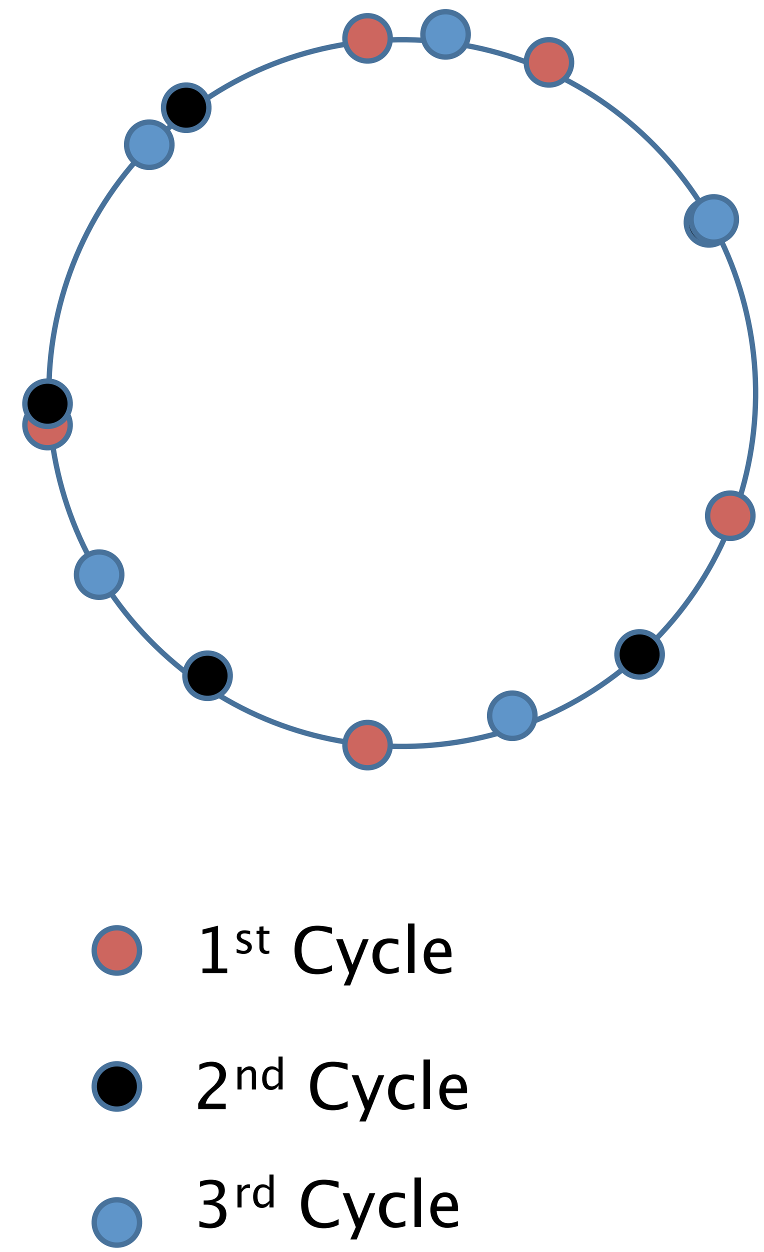

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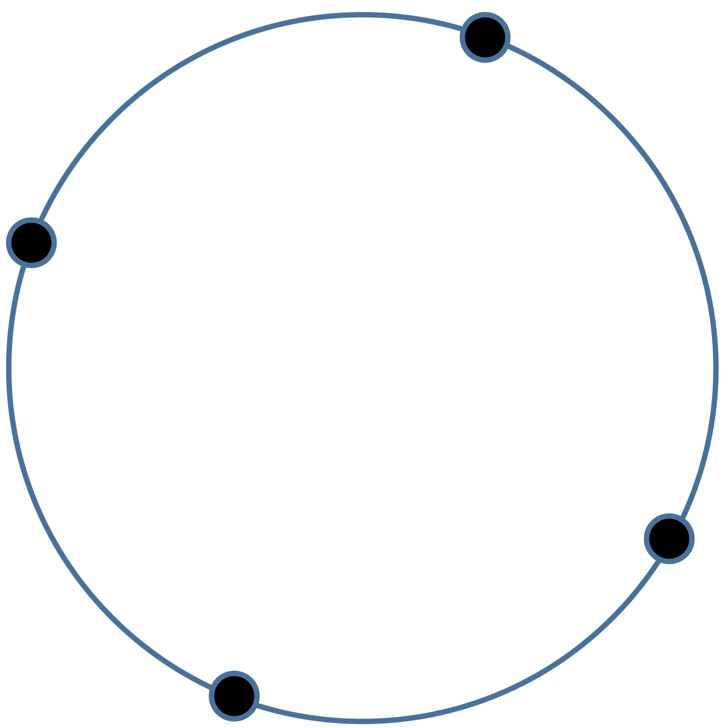

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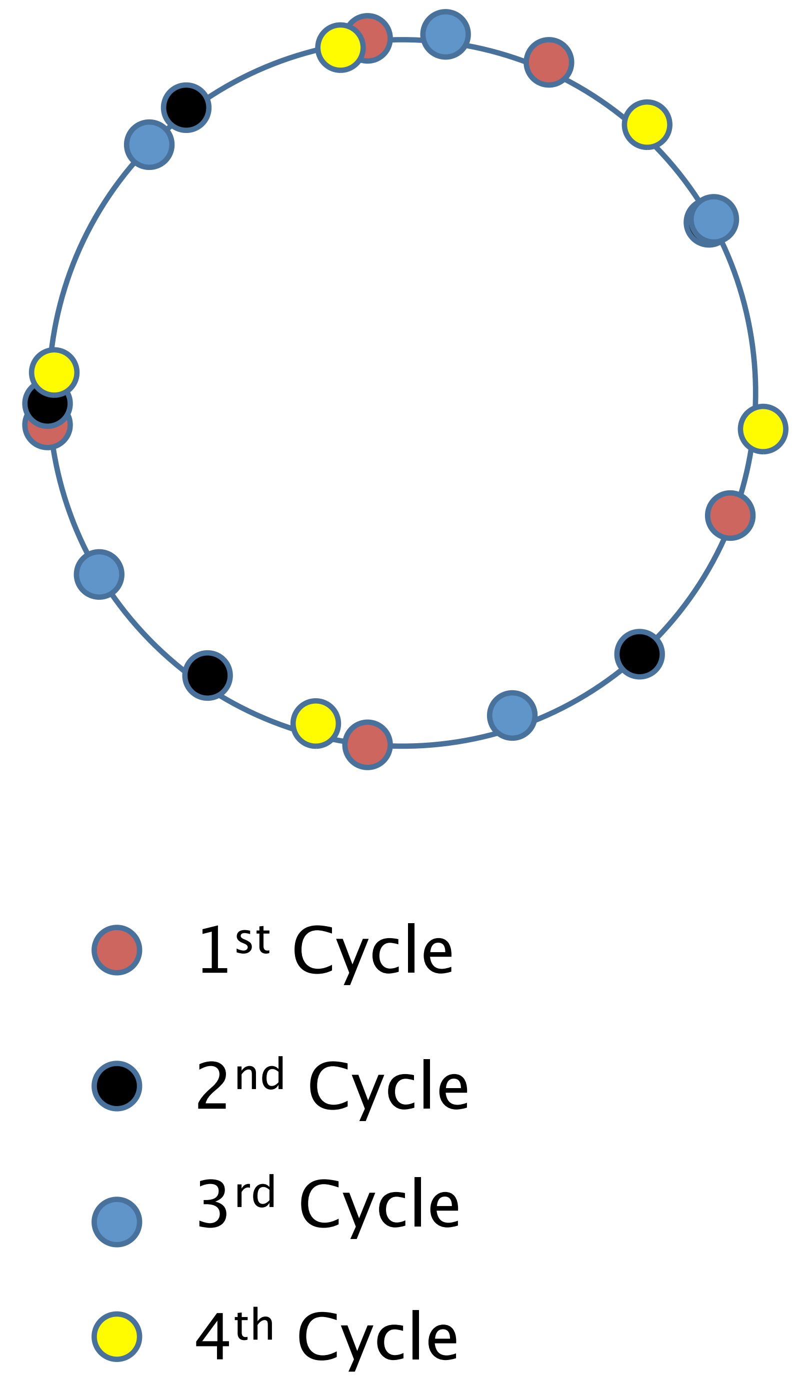

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- [MOC] stands for "Minimal Orbit Closures".
- If T has [MOC], then X splits into distinct T-invariant components (closed orbits) and the T-action on each component is ergodic.

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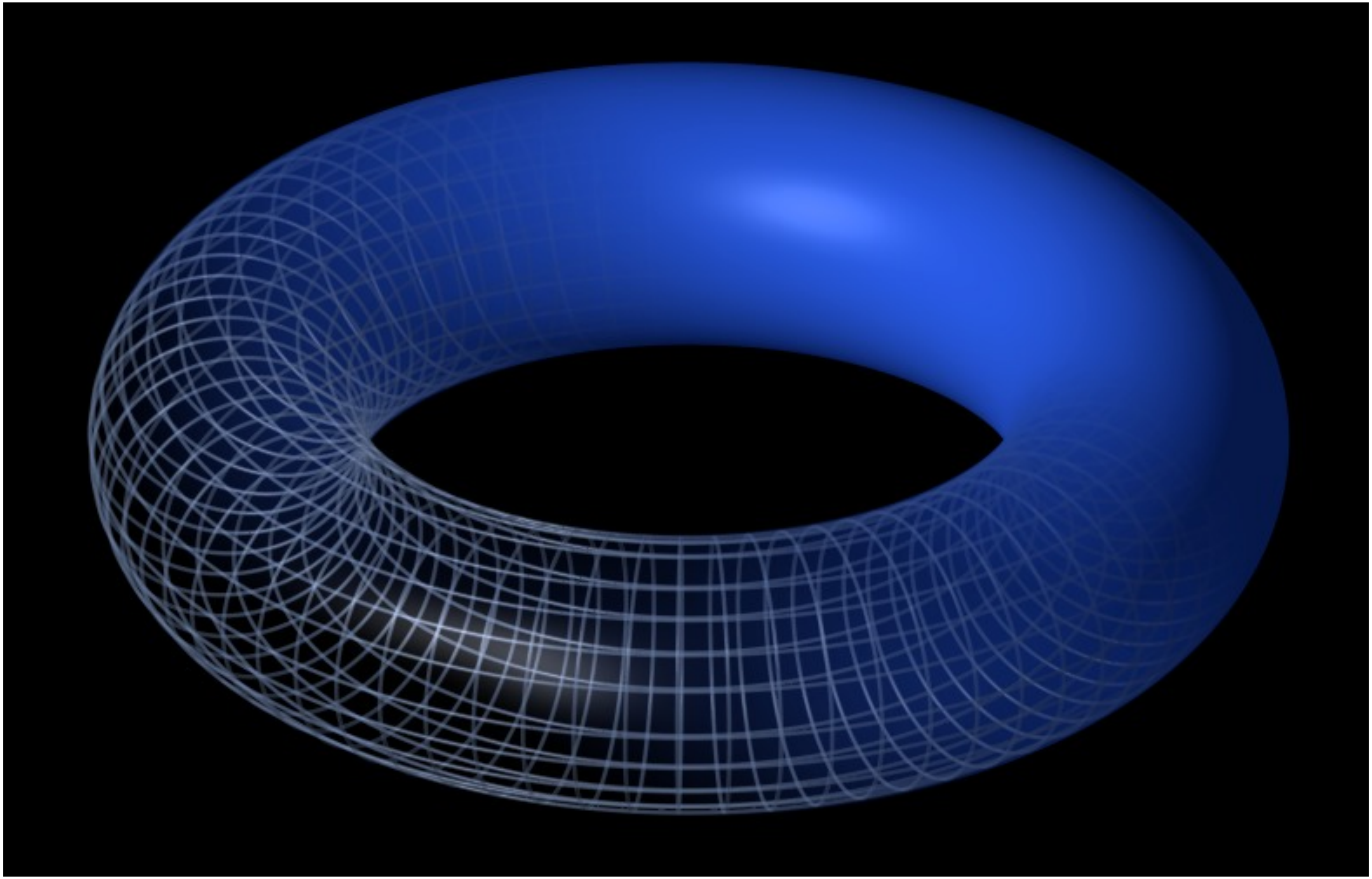
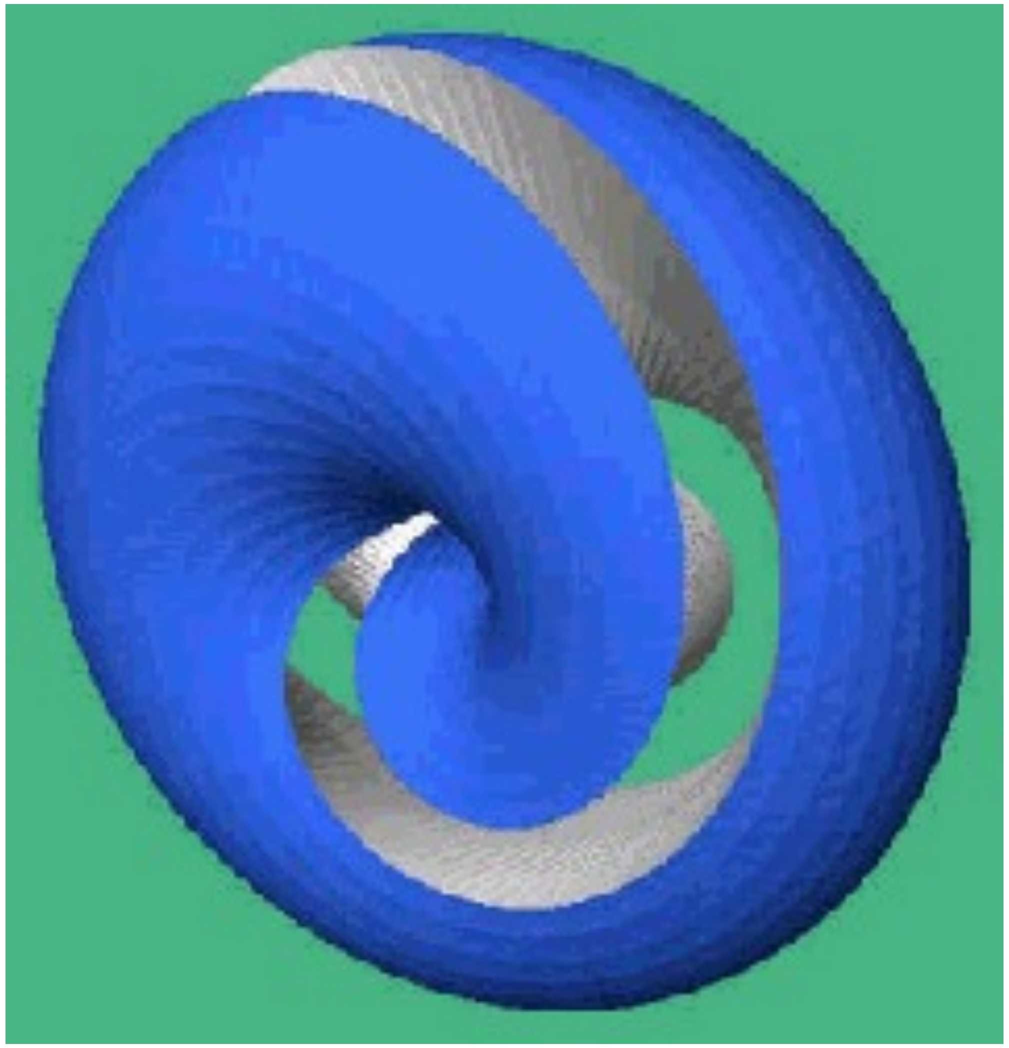
- So no orbit is dense in the torus $S^1 \times S^1$.
- This map T has [MOC].
  (Here we have some pictures.)



Torus after one iteration of T



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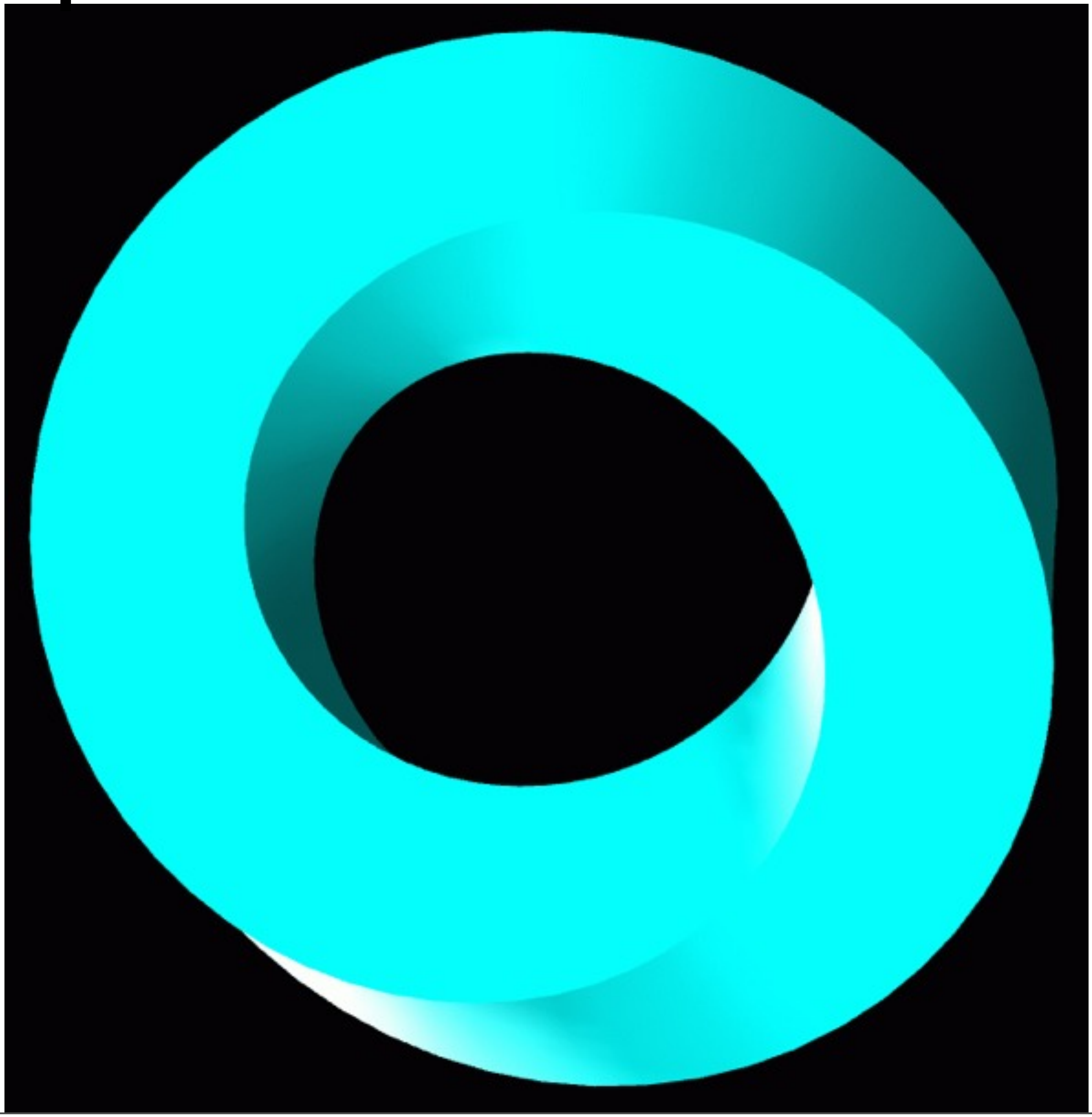
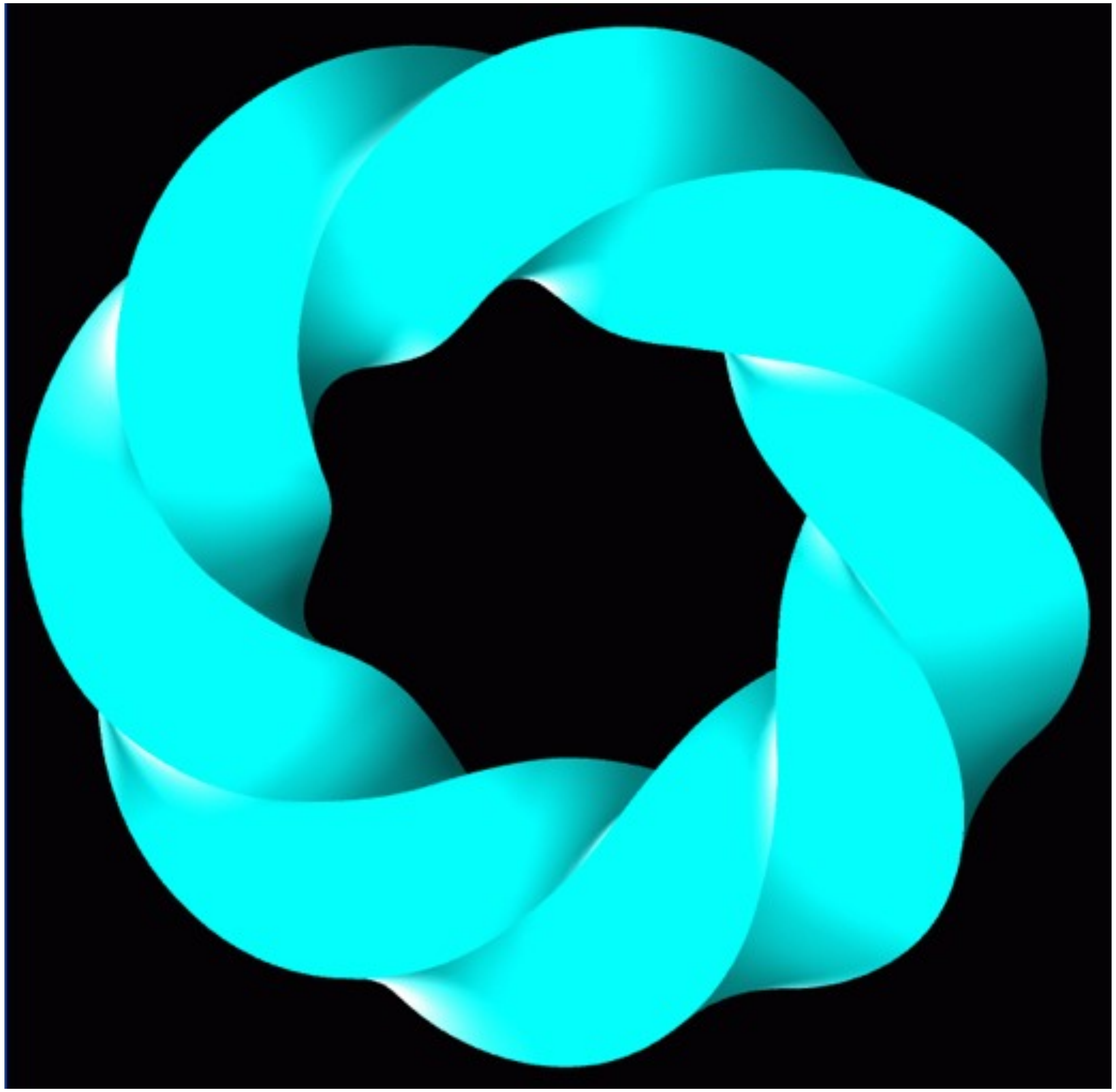


Torus After a few Iteration of



Distality and Orbits

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- If T is distal, is each orbit closure cl{$T^n(x)$} (or cl{$T^t(x)$}) a minimal closed invariant set?
- This is known to be true for any distal action on compact spaces, (for e.g. rotation maps, or the map on torus described before).

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(For e.g. on Lie groups are locally like Euclidean spaces; closed subgroups of the matrix group; $GL(n,R)$=group of $n \times n$ invertible matrices$\approx$ Linear transformations on R^n).