FOUND:

Yet another POINT OF INTERSECTION BETWEEN GEOMETRY & PHYSICS:

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Points of intersection: geometry & physics

* Einstein used Riemann's description of curved shapes & geometry as the language for his theory of gravity.
* This geometry is used to precisely find the path of light rays near the sun or planetary orbits.

Many such examples ....

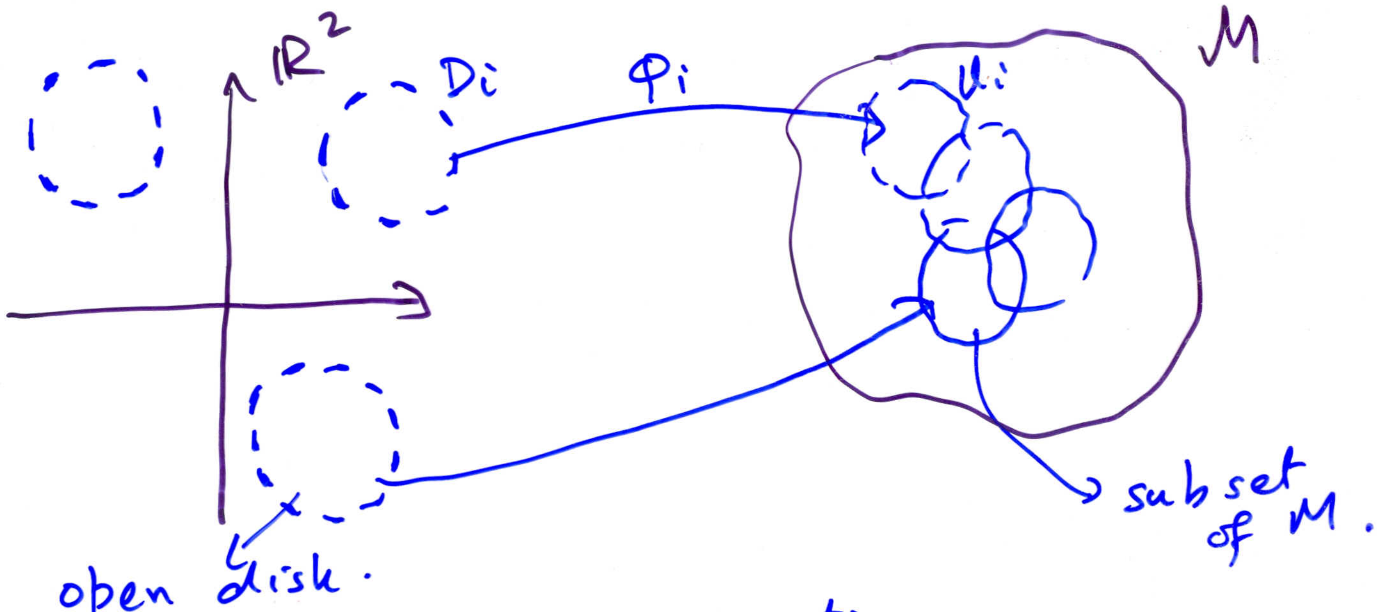
A new point of intersection: Ricci Flow

* Ricci flow is a 'flow through geometries'
* It was used to prove the Thurston geometrization & Poincare conjectures.
* It arises naturally in physics as a Renormalization group flow.

Q What are the implications of this point of intersection?

Ricci flow: defined on a smooth manifold M with Riemannian metric g:

ROUGH IDEA OF A SMOOTH MANIFOLD in 2 dimensions:
(intrinsic definition of a curved surface).



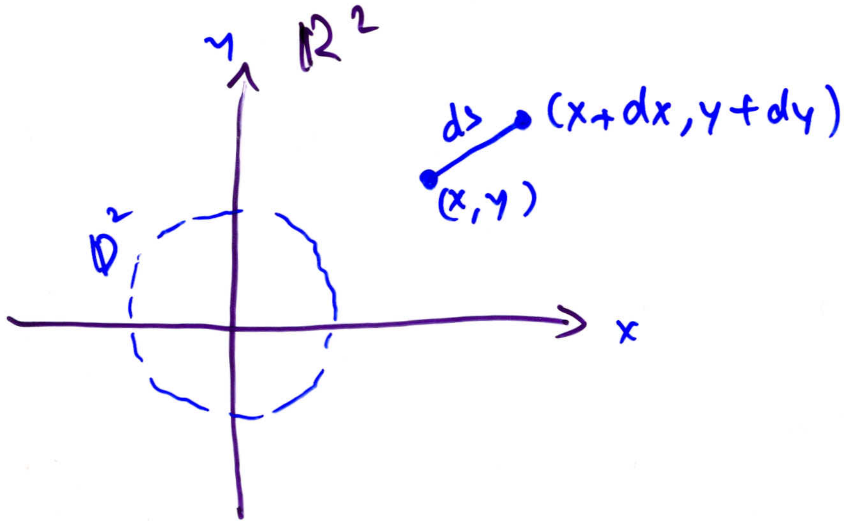
$\varphi_i : D_i \to U_i$ is bijective.

"Take cutouts of open disks in $\mathbb{R}^2$ & glue them to each other in a smooth way to get a curved space" → M "looks locally" like $\mathbb{R}^2$.

An n-dimensional smooth manifold M "looks locally" like $\mathbb{R}^n$.

Riemannian metric on a manifold : +

(this is a rough, very nonrigorous idea of what it is).



$$ds^2 = dx^2 + dy^2 \quad \text{(Euclidean distance).}$$

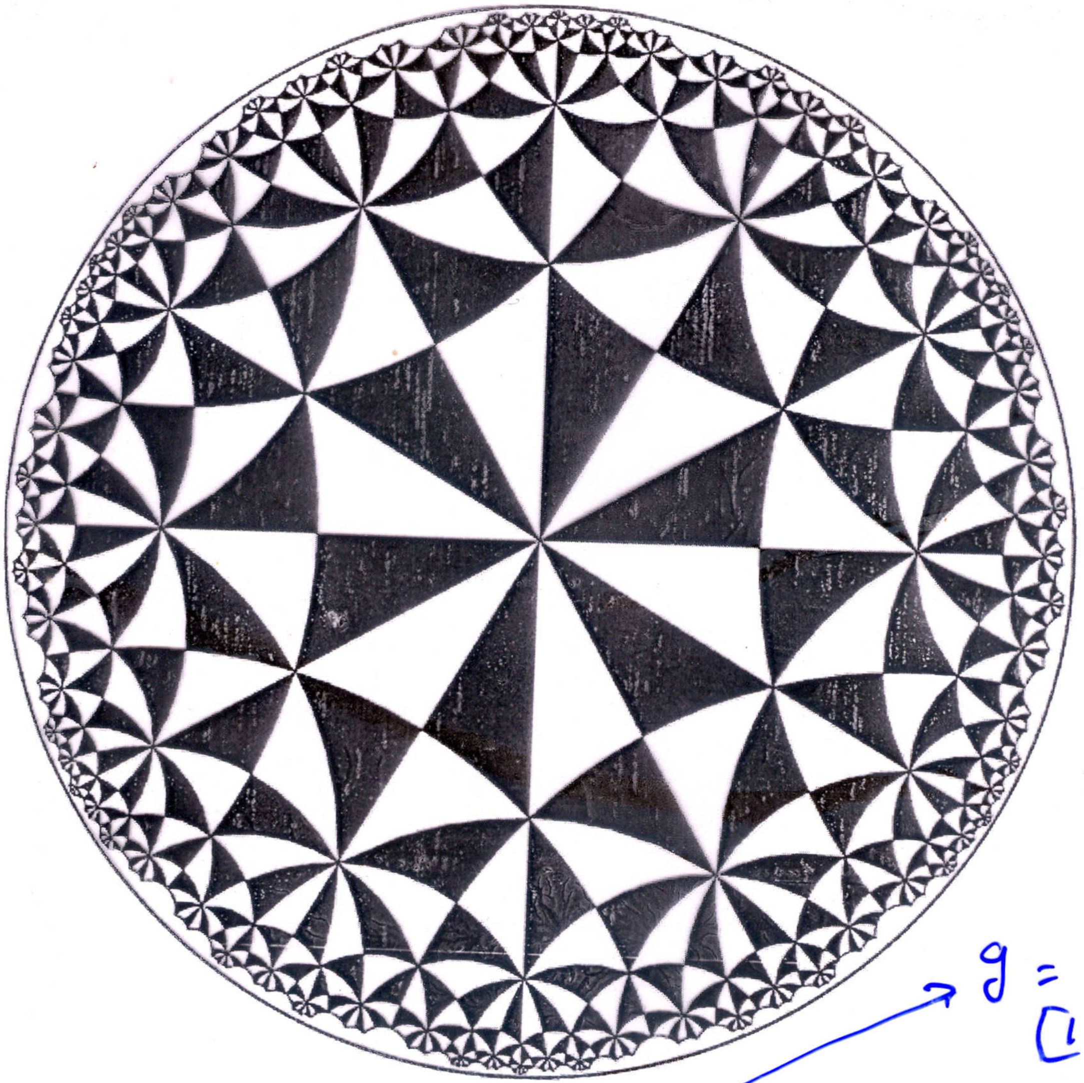
$$= d\bar{x}^T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{g} d\bar{x}$$

$g \longrightarrow$ "Euclidean metric"

(can be used to define areas : $d\mu = dx\,dy$).

Q Can we define other "distance functions" on the plane or its subsets?

Yes! For e.g, take open unit disk $\mathbb{D}^2$.



open unit disk in $\mathbb{R}^2$.

$x^2+y^2<1$

$$g = \frac{g_{Euc.}}{[1-(x^2+y^2)]^2}$$

Hyperbolic metric on unit disk:

$$ds^2 = \frac{1}{(1-[x^2+y^2])^2}\left[dx^2+dy^2\right]$$

Infinitesimal area: $d\mu = \dfrac{dx\,dy}{[1-(x^2+y^2)]^2}$.

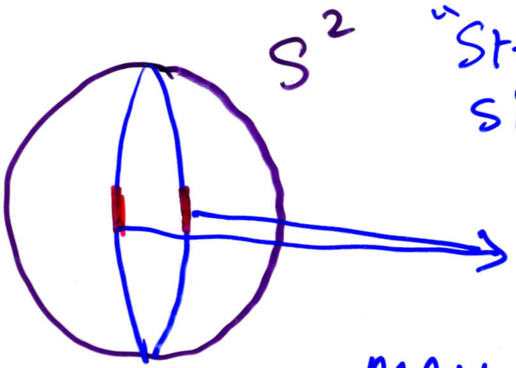
$\Rightarrow$ All "triangles" in above figure have the same area.

METRIC & CURVATURE :

What we called "metric" was related to the distance function. It is also related to "intrinsic curvature" of a manifold.

Manifestations of "intrinsic curvature".

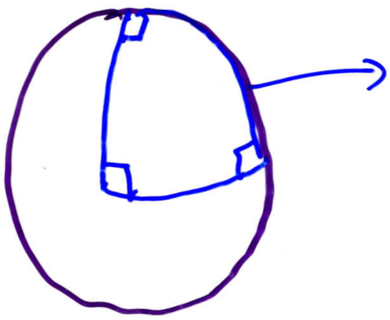
(i)



"Straight lines" = paths of shortest distance.

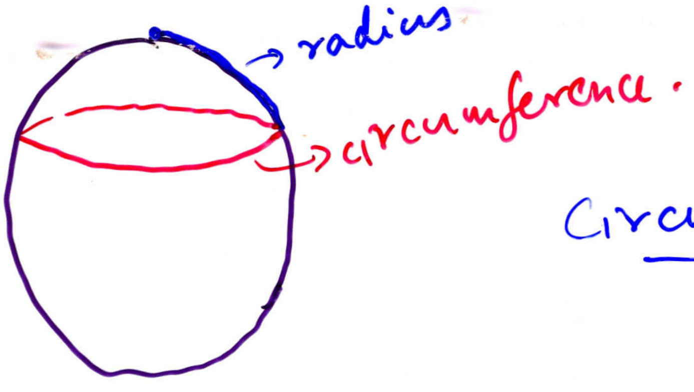
Two nearby initially parallel straight lines may converge or diverge proportional to curvature.

(ii)



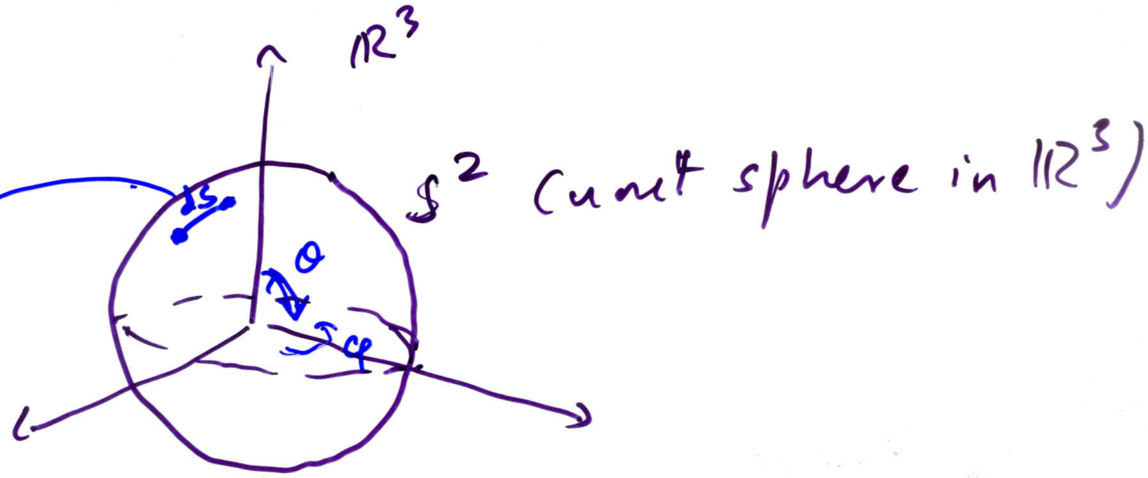
Sum of angles of a triangle $\neq 180°$.

(iii)



$$\frac{\text{Circumference}}{\text{radius}} \neq 2\pi$$

Non-euclidean metrics are "natural" on curved spaces ....



$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$ (follows from Euclidean distance on $\mathbb{R}^3$)

$$ds^2 = d\bar{x}^T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}}_{g} d\bar{x} \quad ; \quad d\bar{x} = \begin{pmatrix} d\theta \\ d\varphi \end{pmatrix}$$

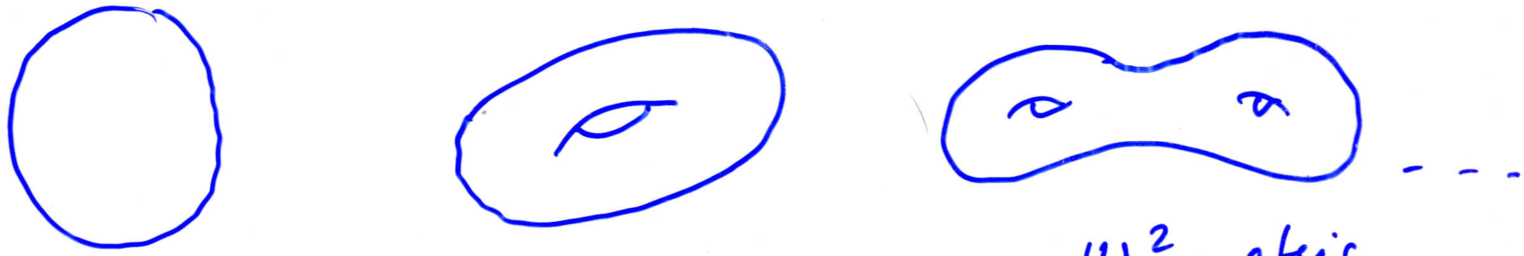
$g \to$ 'sphere metric'

Matrix g in general has components which are real valued functions, it is symmetric & has all eigenvalues positive.
(these properties guarantee that it can be used to define a distance function).
(This matrix is closely related to a Riemannian metric g on a manifold M)

Q// Can we put any arbitrary Riemannian metric on a manifold M? NO!

UNIFORMIZATION THEOREM (surfaces):

Every compact, connected, orientable surface admits one of 3 special metrics:



S^2 metric curvature $= +1$

$\mathbb{R}^2$ metric curv. $= 0$

$\mathbb{H}^2$ metric curv. $= -1$.

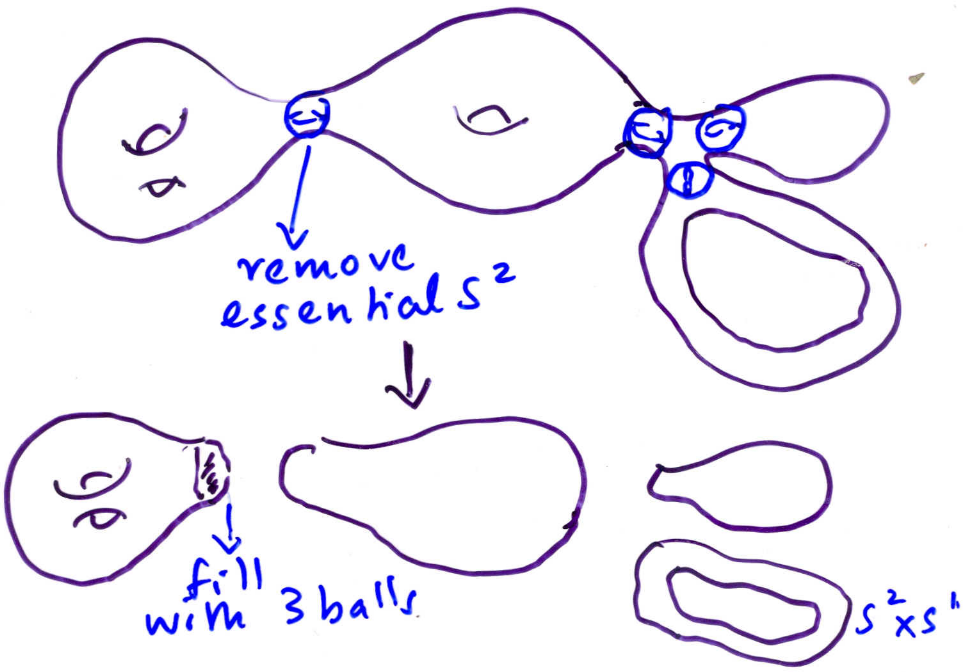
Q// What about 3 dimensions? Are there any special metrics in three dimensions?

NO (really: but Thurston made a more complicated conjecture concerning 8 special metrics ----)

Thurston's geometrization conjecture:

States: If you first do a canonical decomposition of a compact, connected, orientable 3-manifold:

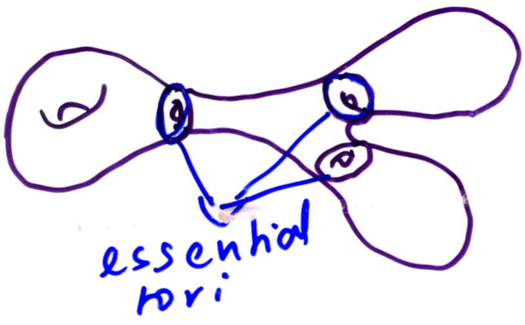
(i)



"PRIME DECOMPOSITION".

"Prime manifolds" unique.

(ii) In each prime manifold, if they exist, remove essential tori:



TORUS DECOMPOSITION.

CONJECTURE.

Resulting manifolds after this decomposition admit one of eight special metrics.

(S^3, $\mathbb{H}^3$, $\mathbb{R}^3$, ...).

(Poincare conjecture follows as a special case).

PROOF OF THURSTON CONJECTURE: (Perelman 2003)

Idea: (R. Hamilton & others)

Start with a partial differential equation on (M, g):

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \qquad \text{"RICCI FLOW"}$$

g_{ij} : ijth component of metric g.

R_{ij} : "Ricci tensor" a measure of curvature of M.

* This partial diff.-eqn. is NONLINEAR.

$$\frac{\partial g_{ij}}{\partial t} \approx \frac{\partial^2 g_{ij}}{\partial x^2} + \cdots,$$

(Parabolic PDE: like a heat equation)

* This equation is a flow of 'geometry' or a flow of curvature.

Q: How was this equation used to prove Thurston's conjecture?

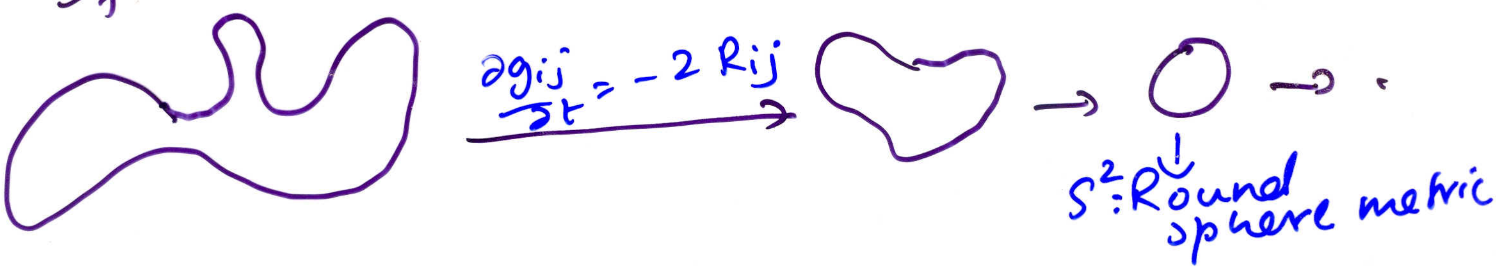
E.g. Consider this flow on a compact connected surface.

(i) Start with any Riemannian metric $g_{ij}(t=0)$ as initial condition.

(ii) See what it flows to.

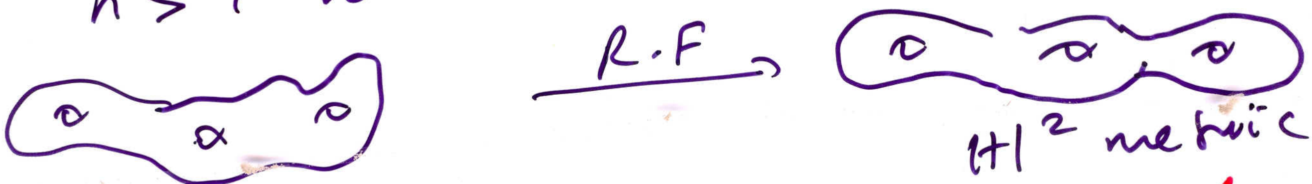
RESULT:

If the surface has no holes, then



If it has a hole:

$n > 1$ holes:



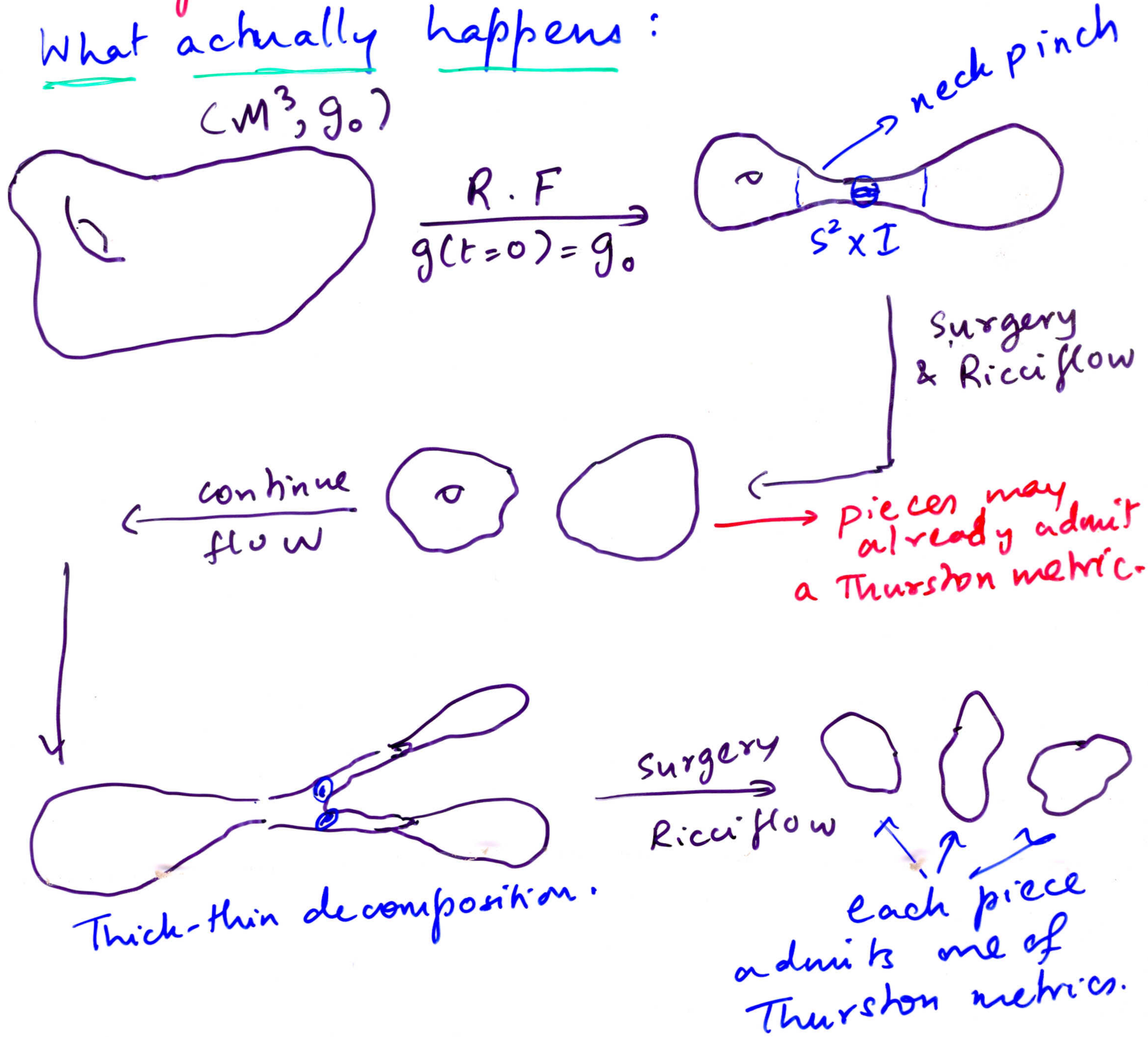
Heat eqn. $\rightarrow$ smooths out initial concentration of heat/temperature
$\rightarrow$ constant temp.

Ricci flow $\rightarrow$ smooths out curvature $\rightarrow$ constant curvature

In 3d, can Ricci flow "implement" Canonical decomposition ??

Yes !! Ricci flow is nonlinear & can lead to singularities.

What actually happens :



Key results important for this: (Perelman)

* Monotonicity of Ricci flow in any dimension on compact M. (no periodic solutions in geometry)

* Complete understanding of all types of singular behaviour in 3 dimensions (compact M).

* Implementation of Ricci flow with surgery.

⇒ Radical new ideas/techniques introduced to the field.

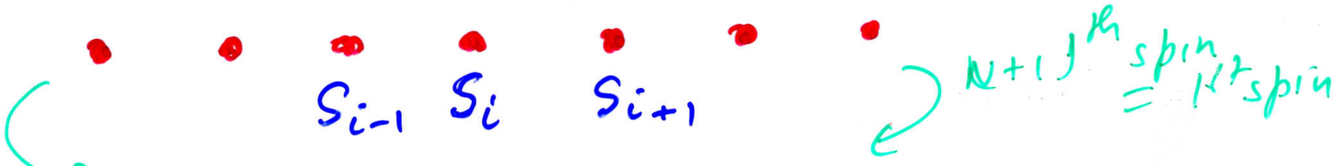
SURPRISINGLY —

Ricci flow also arose in physics (in fact before it was introduced in math). The key results are important for physics as well.

"Ricci flow arises as a renormalization group (RG) flow in physics"

Renormalization group (RG) flow in physics:

Example: Physical system of N "spins" on a one-dim. lattice ; $N+1 \cong 1$



Each lattice has a spin, which has a value $s_i \in \{-1, 1\}$

"Interaction" between spins: This means that changing the value of a spin affects the system. Interaction is only between nearest neighbours.

All experimentally measurable quantities are obtained from "Partition function": $Z = Tr_N\, e^{H}$

where

$$H = K \sum_i s_i s_{i+1} + h \sum_i s_i + CN.$$

(k, h, c) are real constants.

" COUPLING CONSTANTS"

In physics, "coupling constants" measure the strength of an interaction & can be found in an experiment.

Partition function:

$$Z = Tr_N(e^{H}) = \prod_i \frac{1}{2} \sum_{s_i = \pm 1} e^{H\{s_i\}}.$$

RENORMALIZATION GROUP TRANSFORMATION:

Step 1: Tr/Sum over alternate spins in Z:

We get:

$$Z = \prod_i \frac{1}{2} \sum_{s_i = \pm 1} e^{H'\{s_i\} + \underbrace{\cdots\cdots}_{\text{non zero for most systems}}}$$

$H'\{s_i\} = k' \sum s_i s_{i+1} + h' \sum s_i + C' N.$

$H'\{s_i\}$ is of same form as H, except:

$$(k, h, c) \longrightarrow (k', h', C').$$

→ Defined on a lattice with spins $N/2$, & twice the lattice spacing.

Step 2: Rescale lattice lengths so that Lattice after step 1 looks superficially like lattice we started out with.

Do this procedure iteratively many times .... to a good approximation,

$\mathcal{H} \xrightarrow[\text{iteration}]{1^{st}} \mathcal{H}' \xrightarrow{2^{nd}} \mathcal{H}'' \xrightarrow{3^{rd}}$ & so on.

$(k, h, C) \xrightarrow{1^{st}} (k', h', C') \xrightarrow{2^{nd}} (k'', h'', C'') \rightarrow$

Lattice spacing 1 unit $\rightarrow$ 2 units $\rightarrow$ 3 units $\rightarrow$

"SUCCESSIVE ZOOMING OUT" in our description of the system.

There is no natural inverse for this procedure.

In most physical systems....

* We go in this way to a simpler looking physical system after many iterations.

* Expectation:

Physical system description $\xrightarrow[\text{RG transformations}]{\text{many}}$ Simpler physical system

is irreversible.

(No general proof of this).

RG transformations : continuum physical systems.

Lattice $\longrightarrow$ some smooth manifold like $\mathbb{R}^n$

$\{s_i\} \longrightarrow \Phi(x)$ "fields"

$H\{s_i\} \longrightarrow \mathcal{L}\{\Phi(x)\}$

$$Z = \mathrm{Tr}_N e^H \longrightarrow Z = \int_{|x| \geq a} [\mathcal{D}\Phi(x)] e^{-\mathcal{S}[\Phi]}$$

An example of $\mathcal{L}[\Phi]$:

$$\mathcal{S}[\Phi] = \int_{\mathbb{R}^n} d^n x \left[\sum_i (\partial_i \Phi)(\partial^i \Phi) - m^2 \Phi^2 + \lambda^4 \Phi^4 \right]$$

where (m^2, λ) are "coupling constants" measuring strength of interactions.

Discrete increase in lattice spacing $\longrightarrow$ continuous change in length scales.

$x \longrightarrow bx$ $b \geq 1$ is a real parameter.

$(k, h, c) \to (k', h', c') \to (k'', h'', c'') \leftarrow$ $\longrightarrow$ $m^2(t), \lambda(t)$ where $t = (\ln b)$.

t : RG "flow" parameter.

In the continuum, we can make RG transformations continuous.

RG flow parameter t: Represents by how much we are "zooming out" in our description of the physical system.

* Coupling "constants" $m^2(t)$, $\lambda(t)$ flow with t ....

* Experimentally seen! Strength of interaction changes at different length scales.

* Flow of $m^2(t)$, $\lambda(t)$ etc. can be given in terms of ODEs/PDE's.

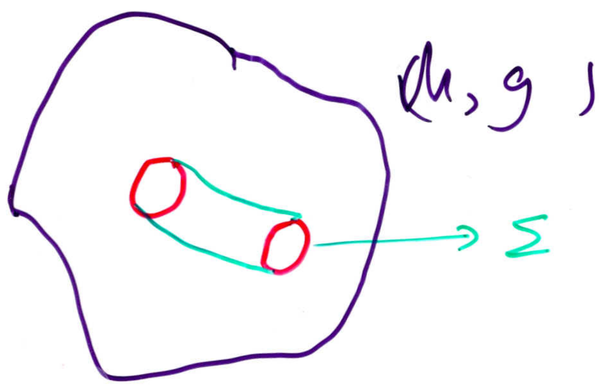
$$\frac{dm^2}{dt} = -\beta(t, m^2, \lambda) \quad \text{etc.}$$

THIS IS CALLED A

"RENORMALIZATION GROUP" (RG) FLOW

Ricci flow as an RG flow:

Setting: String theory
(string is fundamental unit of matter in this theory)



$X : \Sigma \to M$

summation sign

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma\, \delta^{\alpha\beta} \sum_{i,j,\alpha,\beta} g_{ij}(X)\left(\frac{\partial X^i}{\partial \sigma^\alpha}\right)\left(\frac{\partial X^j}{\partial \sigma^\beta}\right) + \cdots$$

some constant.

g_{ij}: ijth element of metric g.
g is a "coupling constant" measuring strength of an interaction. Flows under an RG transformation.

RG flow:

$$\frac{\partial g_{ij}}{\partial t} = -\alpha' R_{ij} + \underbrace{\cdots}_{O(\alpha'^2)}$$

some (small) constant in string theory.

Q What are the questions of interest to physicists in this system?

→ Ricci flow to first order in α'!

Importance of Ricci flow results for physics:

(i) In a certain approximation, solutions to this RG flow tell us how the propagating string (matter) changes the geometry of the space in which it propagates. Solutions to Ricci flow, singular behaviour under Ricci flow are all of interest.

(ii) Monotonicity of Ricci flow is very important. RG transformation suggests that in most physical systems, RG flow is "irreversible". Perelman's result for Ricci flow confirms this.

Q// Can tools & techniques from physics give insight to questions in Ricci flow?

The MOST important area where this may happen:

Ricci flow on non compact manifolds:

Some results so far:

(i) Ricci flow on asymptotically flat manifolds is monotonic. (Oliynyk, Woolgar, S.V)

(ii) Extension of Perelman's results to other geometric flows. (Oliynyk, Woolgar, S.V)

(iii) Linear stability of $\mathbb{H}^n$ under Ricci flow. (S.V)

!

Most results obtained by using techniques of physicists working in gravitation & relativity.

Many more tractable questions!