THE ABEL-JACOBI ISOMORPHISM ON ONE CYCLES ON THE MODULI SPACE OF VECTOR BUNDLES WITH TRIVIAL DETERMINANT ON A CURVE

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Abstract. We consider the moduli space $\mathcal{SU}^s_C(2,\mathcal{O}_C)$ of rank 2 stable vector bundles with trivial determinant on a smooth projective curve C of genus g. We show that the Abel-Jacobi map on the rational Chow group $CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}$ of one cycles which are homologous to zero, is an isomorphism onto the bottom weight intermediate Jacobian, which is identified with the Jacobian $Jac(C)\otimes\mathbb{Q}$.

Contents

1. Introduction

Suppose C is a smooth connected complex projective curve of genus g. For a line bundle L on C consider the moduli space $\mathcal{SU}_C(r,L)$ of semi-stable vector bundles of rank r and of fixed determinant L on C. In this paper we assume that either $L:=\mathcal{O}_C$ or $L:=\mathcal{O}_C(x)$ for some point $x\in C$. When $L=\mathcal{O}_C$ the moduli space $\mathcal{SU}_C(r,\mathcal{O}_C)$ is a (singular) normal projective variety and when $L=\mathcal{O}_C(x)$ the moduli space $\mathcal{SU}_C(r,\mathcal{O}(x))$ is a smooth projective variety. The space of stable bundles $\mathcal{SU}^s_C(r,\mathcal{O}_C)\subset\mathcal{SU}_C(r,\mathcal{O}_C)$ forms a smooth quasi-projective variety. Both the moduli spaces have dimension equal to $(r^2-1)(g-1)$.

It is known that these moduli spaces are unirational [Se3]. In fact, the moduli spaces $\mathcal{SU}_C(r,\mathcal{O}(-x))$ and $\mathcal{SU}^s_C(r,\mathcal{O}_C)$ are Fano manifolds [Ra], [Be]. This implies that the rational Chow group of zero cycles is trivial, i.e., $CH_0(\mathcal{SU}_C(r,L))\otimes\mathbb{Q}\simeq\mathbb{Q}$. More generally, when X is any smooth unirational variety of dimension equal to n then it is

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well-known that the Hodge groups $H^p(X,\Omega^q_X) = 0$ whenever $p = 0$. In this case the intermediate Jacobian $IJ^p(X) := \frac{H^{2p-1}(X,\mathbb{C})}{F^p+H^{2p-1}(X,\mathbb{Z})}$ for $p = 2, n-1$ is an abelian variety. It is of interest to show the *weak representability* via the Abel-Jacobi map

$$CH^p(X)_{hom} \xrightarrow{AJ^p} IJ^p(X)$$

for $p = 2, n-1$ and also determine the abelian variety in terms of the geometry of X. Some examples of Fano threefolds F were shown to have *weakly representable* $CH_1(F)$ by Bloch and Murre [Bl-Mr]. When $X = \mathcal{SU}_C(2,\mathcal{O}(x))$, we have

$$\begin{aligned}
CH^1(\mathcal{SU}_C(r,\mathcal{O}(x)))\otimes\mathbb{Q} &\simeq \mathbb{Q} \text{ [Ra]}\\
CH^2(\mathcal{SU}_C(2,\mathcal{O}(x)))\otimes\mathbb{Q} &\simeq CH_0(C)\otimes\mathbb{Q},\ g=2\\
&\simeq CH_0(C)\otimes\mathbb{Q}\oplus\mathbb{Q},\ g>2 \text{ [Ba-Kg-Ne]}\\
CH_1(\mathcal{SU}_C(2,\mathcal{O}(x)))\otimes\mathbb{Q} &\simeq CH_0(C)\otimes\mathbb{Q} \text{ [Ch-Hw].}
\end{aligned}$$

Set $n := (r^2-1)(g-1)$ and $AJ_1 := AJ^{n-1}$. Since the moduli space $\mathcal{SU}_C(2,\mathcal{O}_C)$ is a singular variety, we do not have good Abel-Jacobi maps and perhaps other Chow/cohomology theories should be considered. We do not look at these theories in this paper. Instead we consider the smooth variety $\mathcal{SU}^s_C(2,\mathcal{O}_C)$ and whose cohomology groups are known to have a mixed Hodge structure. We consider the Abel-Jacobi maps which take values in the intermediate Jacobian of the bottom weight cohomologies. Using the results of Arapura-Sastry [Ar-Sa] in the trivial determinant case, we know that the degree 3 rational cohomology group has a pure Hodge structure of weight 3 and the intermediate Jacobian IJ^2 is identified with the Jacobian variety $Jac(C)\otimes\mathbb{Q}$. In order to study the Abel-Jacobi map AJ_1, we firstly need to identify the target group. This is done by showing an isomorphism $Jac(C)\otimes\mathbb{Q} \simeq IJ(W_{2n-3}H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C)))$ given by the Lefschetz operator, see Lemma 3.7.

With notations as above, we show:

Theorem 1.1. *Suppose $g \geq 3$, $r \geq 2$ and fix any point $x \in C$. Then the Abel-Jacobi map on the rational Chow group of one cycles homologous to zero*

$$CH_1(\mathcal{SU}_C(r,\mathcal{O}(x)))_{hom}\otimes\mathbb{Q} \xrightarrow{AJ_1} Jac(C)\otimes\mathbb{Q} \tag{1}$$

is always surjective. The same assertion is true on the non-compact smooth variety $\mathcal{SU}^s_C(2,\mathcal{O}_C)$, whenever $g \geq 4$. Furthermore, in this case, the Abel-Jacobi map extends to isomorphisms

$$CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \simeq Jac(C)\otimes\mathbb{Q},$$
$$CH_1(\mathcal{SU}_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \simeq Jac(C)\otimes\mathbb{Q}.$$

Along the way, we also look at the codimension two cycles on the variety $\mathcal{SU}^s_C(2,\mathcal{O}_C)$ and prove the isomorphism

$$CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \simeq Jac(C)\otimes\mathbb{Q}$$

whenever $g \geq 3$, see Lemma 3.11. The proof of Theorem 1.1 is by combining the methods of [Ch-Hw] and the Hecke correspondence employed in [Ar-Sa]. The first assertion uses the correspondence cycle $c_2(U)$ (where U is a universal Poincaré bundle [Na-Ra]) between the Chow groups of 0-cycles on C and the codimension 2 cycles on $\mathcal{SU}_C(2, \mathcal{O}(x))$. Together with the isomorphism of the Lefschetz operator on $H^3(\mathcal{SU}_C(r, \mathcal{O}(x)), \mathbb{Q})$, we conclude the surjectivity of the Abel-Jacobi map on one cycles. In the case of the non compact smooth variety $\mathcal{SU}^s_C(r, \mathcal{O}_C)$, a Hecke correspondence was used in [Ar-Sa] for computing the low degree cohomology groups in terms of the cohomology of $\mathcal{SU}_C(r, \mathcal{O}(-x))$. We use this correspondence to relate the Chow group of one cycles and codimension two cycles, of these two spaces when $r = 2$.

The Hard Lefschetz isomorphism plays a crucial role in identifying the target group of the Abel-Jacobi map AJ^s_1 on one cycles. In general, the Hard Lefschetz isomorphism is not true for open smooth varieties. In our situation, we show that the Hard Lefschetz isomorphism holds for the bottom weight cohomologies of $\mathcal{SU}^s_C(2, \mathcal{O}_C)$ in odd degree, see Lemma 3.7. This is proved using the desingularisation $\mathcal{S}$ of $\mathcal{SU}_C(2, \mathcal{O}_C)$ constructed by Seshadri ([Se2]) and using the description of the exceptional locus. The exceptional locus has a stratification given by the rank of a conic bundle [Ba], [Ba-Se]. The computation of the low degree cohomology of $\mathcal{S}$ in [Ba-Se] used the Thom-Gysin sequences for the stratification. We also use similar long exact seqences of Borel-Moore homologies for the stratification of $\mathcal{S}$. A closer analysis reveals that the bottom weight Borel-Moore homologies $W_{-n+i}H_{n-i}$ in odd degree of the exceptional divisor are zero. This suffices to conclude the Hard Lefschetz isomorphism on the bottom weight odd degree cohomologies of $\mathcal{SU}^s_C(2, \mathcal{O}_C)$. In particular we identify the target group of AJ^s_1 to be $IJ(W_{2n-3}H^{2n-3}(\mathcal{SU}_C(2, \mathcal{O}_C))) \simeq Jac(C) \otimes \mathbb{Q}$.

To conclude the theorem, we need to find a minimal generating set of one cycles on $\mathcal{SU}_C(2, \mathcal{O}_C)$. The Hecke curves [Na-Ra2] are minimal rational curves on the moduli space $\mathcal{SU}_C(2, \mathcal{O}_C)$. A variant of a theorem of Kollar [Ko, Proposition 3.13.3] on Chow generation, is proved by J-M. Hwang and he observed that the Hecke curves generate the rational Chow group of one cycles on $\mathcal{SU}_C(2, \mathcal{O}_C)$, see Proposition 4.1, Corollary 4.2. This gives a surjective map

$$Jac(C) \otimes \mathbb{Q} \to CH_1(\mathcal{SU}^s_C(2, \mathcal{O}_C))_{hom} \otimes \mathbb{Q}.$$

Together with the Abel-Jacobi surjectivity, we conclude our main theorem.

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2. Abel-Jacobi surjectivity for one cycles on $\mathcal{SU}_C(r,\mathcal{O}(x))$

Suppose C is a smooth connected projective curve defined over the complex numbers of genus g. Fix a point $x \in C$ and consider the moduli space $\mathcal{SU}_C(r,\mathcal{O}(x))$ of stable vector bundles of rank r and fixed determinant $\mathcal{O}(x)$ on C. Atiyah and Bott [At-Bo] have described the generators of the cohomology ring $H^*(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q})$ in terms of the characteristic classes of the Poincaré bundle. Since we are concerned only with certain cohomologies in low degree, we will recall the generators in low degrees. We also note that the moduli space $\mathcal{SU}_C(r,\mathcal{O}(x))$ is isomorphic to $\mathcal{SU}_C(r,\mathcal{O}(-x))$ given by $E \mapsto E^*$, the dual of E. In the next section we will consider the moduli space $\mathcal{SU}_C(r,\mathcal{O}(-x))$ and relate it with the results from this section.

Fix a Poincaré bundle $U \to C \times \mathcal{SU}_C(r,\mathcal{O}(x))$ and denote the i-th Chern class of U by $c_i(U) \in H^{2i}(C \times \mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q})$. Denote the two projections by

$$p_1 : C \times \mathcal{SU}_C(r,\mathcal{O}(x)) \to C,\ p_2 : C \times \mathcal{SU}_C(r,\mathcal{O}(x)) \to \mathcal{SU}_C(r,\mathcal{O}(x)).$$

The cycle $c_i(U)$ acts as a correspondence between the cohomology of C and the cohomology of $\mathcal{SU}_C(r,\mathcal{O}(x))$. More precisely, there are homomorphisms:

$$H^k(C,\mathbb{Q}) \xrightarrow{p_1^*} H^k(C \times \mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q}) \xrightarrow{\cup c_i(U)} H^{k+2i}(C \times \mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q})$$
$$H^{k+2i}(C \times \mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q}) \xrightarrow{p_{2*}} H^{k+2i-2}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q}).$$

The composition $\Gamma_{c_i(U)} := p_{2*} \circ \cup c_i(U) \circ p_1^*$ is called the correspondence defined by the cycle $c_i(U)$.

Theorem 2.1. *The correspondence*

$$\Gamma_{c_2(U)} : H^1(C,\mathbb{Q}) \longrightarrow H^3(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q})$$

is an isomorphism, for $r \geq 2, g \geq 3$.

Proof. See [Na-Ra, Theorem 3] together with [Ra, Section 4 and Lemma 2.1]. The isomorphism is actually with integral coefficients. □

Fix an ample line bundle $\mathcal{O}(1)$ on the moduli space $\mathcal{SU}_C(r,\mathcal{O}(x))$. Denote its class $H := c_1(\mathcal{O}(1) \in H^2(\mathcal{SU}_C(r,\mathcal{O}(x),\mathbb{Q})$. Set $n := \dim \mathcal{SU}_C(r,\mathcal{O}(x))$.

Corollary 2.2. *The composition*

$$H^1(C,\mathbb{Q}) \xrightarrow{\Gamma_{c_2(U)}} H^3(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q}) \xrightarrow{\cup H^{n-3}} H^{2n-3}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Q})$$

is an isomorphism of pure Hodge structures.

Proof. This follows from Theorem 2.1 and the Hard Lefschetz theorem. □

Corollary 2.3. *There is an isomorphism of the intermediate Jacobians*

$$Jac(C) \longrightarrow IJ_1 := \frac{H^{2n-3}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{C})}{F^3 + H^{2n-3}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Z})}$$

induced by the composed morphism $H^{n-3} \circ \Gamma_{c_2(U)}$.

We now want to show that the isomorphism on the intermediate Jacobians are compatible with intersections and correspondences on the rational Chow groups. More precisely, consider the classes $c_2(U) \in CH^2(\mathcal{SU}_C(\mathcal{O}(x))$ and $H \in CH^1(\mathcal{SU}_C(r,\mathcal{O}(x))$. Set $\Gamma^{CH}_{c_i(U)} := p_{2*} \circ \cap c_i(U) \circ p_1^*$, where

$$CH^1(C) \otimes \mathbb{Q} \xrightarrow{p_1^*} CH^1(C \times \mathcal{SU}_C(r,\mathcal{O}(x))) \otimes \mathbb{Q} \xrightarrow{\cap c_2(U)} CH^3(C \times \mathcal{SU}_C(r,\mathcal{O}(x))) \otimes \mathbb{Q}$$
$$CH^3(C \times \mathcal{SU}_C(r,\mathcal{O}(x))) \otimes \mathbb{Q} \xrightarrow{p_{2*}} CH^2(\mathcal{SU}_C(r,\mathcal{O}(x))) \otimes \mathbb{Q}.$$

We consider the correspondence

$$\Gamma^{CH}_{c_2(U)} : CH^1(C) \otimes \mathbb{Q} \longrightarrow CH^2(\mathcal{SU}_C(r,\mathcal{O}(x))) \otimes \mathbb{Q}.$$

This restricts to a correspondence on the subgroup of cycles homologous to zero:

$$\Gamma^{CH}_{c_2(U)} : CH^1(C)_{hom} \otimes \mathbb{Q} \longrightarrow CH^2(\mathcal{SU}_C(r,\mathcal{O}(x)))_{hom} \otimes \mathbb{Q}.$$

Furthermore, there is a composition of morphisms:

$$(2)\quad CH^1(C)_{hom} \otimes \mathbb{Q} \xrightarrow{\Gamma^{CH}_{c_2(U)}} CH^2(\mathcal{SU}_C(r,\mathcal{O}(x)))_{hom} \otimes \mathbb{Q} \xrightarrow{\cap H^{n-3}} CH_1(\mathcal{SU}_C(r,\mathcal{O}(x)))_{hom} \otimes \mathbb{Q}.$$

Set $\psi := \cap H^{n-3} \circ \Gamma^{CH}_{c_2(U)}$.

We recall the exact sequence which relates the intermediate Jacobian with the Deligne cohomology group [Es-Vi]:

$$(3)\qquad 0 \longrightarrow IJ_1 \to H^{2n-2}_{\mathcal{D}}(\mathcal{SU}_C(r,\mathcal{O}(x),\mathbb{Z}(n-1)) \longrightarrow Hodge^{n-1} \longrightarrow 0.$$

Here $Hodge^{n-1} := Image(H^{2n-2}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Z}) \to H^{2n-2}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{C}) \cap H^{n-1,n-1}$.

Recall the Abel-Jacobi maps:

$$CH^1(C)_{hom} \xrightarrow{AJ_C} Jac(C),$$
$$CH_1(\mathcal{SU}_C(r,\mathcal{O}(x)))_{hom} \xrightarrow{AJ_1} IJ_1.$$

Lemma 2.4. *The map ψ defined in* (2) *is compatible with the Abel-Jacobi maps AJ_C and AJ_1. In other words, the following diagram commutes:*

$$\begin{array}{ccc} CH^1(C)_{hom} \otimes \mathbb{Q} & \xrightarrow{\psi} & CH_1(\mathcal{SU}_C(r,\mathcal{O}(x))_{hom} \otimes \mathbb{Q} \\ \downarrow AJ_C & & \downarrow AJ_1 \\ Jac(C) \otimes \mathbb{Q} & \xrightarrow{\simeq} & IJ_1 \otimes \mathbb{Q}. \end{array}$$

Proof. There are cycle class maps

$$CH_1(\mathcal{SU}_C(r,\mathcal{O}(x))) \to H^{2n-2}_{\mathcal{D}}(\mathcal{SU}_C(r,\mathcal{O}(x)),\mathbb{Z}(n-1))$$
$$CH^1(C) \to H^2_{\mathcal{D}}(C,\mathbb{Z}(1))$$

which induces the Abel -Jacobi map on the subgroups of cycles which are homologous to zero (see [Es-Vi]). In other words, using the exact sequence (3), we note that the Deligne cycle class map on the subgroup of cycles homologous to zero factors via the intermediate Jacobian and this map is the same as the Abel-Jacobi map. Furthermore, the cycle class map into the Deligne cohomology is compatible with correspondences and intersection products on the Chow groups. This implies that the above diagram in the statement of the lemma is commutative. The isomorphism on the last row of the commutative diagram is given by Corollary 2.3. □

Corollary 2.5. *The Abel-Jacobi map*

$$AJ_1 : CH_1(\mathcal{SU}_C(r, \mathcal{O}(x))_{hom} \otimes \mathbb{Q} \longrightarrow IJ_1 \otimes \mathbb{Q}$$

is surjective and is a splitting, i.e., the inverse AJ_1^{-1} is well-defined and is injective.

Proof. Use the isomorphism in the last row of the commutative diagram of Lemma 2.4 to obtain the surjectivity of AJ_1 and a splitting. □

We know by Corollary 2.5 that the Abel-Jacobi map AJ_1 is surjective and a splitting. To show that the Abel-Jacobi map is actually an isomorphism, it suffices to show that the the one-cycles on the moduli space is generated by cycles parametrised by the Jacobian $Jac(C)$. In other words there should be a surjective map

$$Jac(C)) \otimes \mathbb{Q} \to CH_1(\mathcal{SU}_C(r, \mathcal{O}(x)))_{hom} \otimes \mathbb{Q}.$$

We will see in §4 how to conclude the Abel-Jacobi isomorphism in the case $r = 2$ and $L = \mathcal{O}_C$, under this assumption.

3. Abel-Jacobi surjectivity for one cycles on $\mathcal{SU}^s_C(2, \mathcal{O}_C)$

We recall the *Hecke correspondence* used by Arapura-Sastry [Ar-Sa] to study the cohomology of the moduli space $\mathcal{SU}^s_C(r, \mathcal{O}_C)$.

As in the previous section, fix a point $x \in C$. Consider the moduli space $\mathcal{SU}_C(r, \mathcal{O}(-x))$ which is a smooth projective variety. Then there exist a Poincaré bundle $\mathcal{P} \longrightarrow C \times \mathcal{SU}_C(r, \mathcal{O}(-x))$ and let $\mathcal{P}_x$ denote the restriction of $\mathcal{P}$ to $\{x\} \times \mathcal{SU}_C(r, \mathcal{O}(-x))$. Denote the projectivisation $\mathbb{P} := \mathbb{P}(\mathcal{P}_x)$ and $\pi : \mathbb{P} \longrightarrow \mathcal{SU}_C(r, \mathcal{O}(-x))$ be the projection. There is a universal exact sequence on $C \times \mathbb{P}$:

$$0 \longrightarrow (1 \times \pi)^* \mathcal{P} \longrightarrow \mathcal{V} \longrightarrow \mathcal{T}_0 \longrightarrow 0 \tag{4}$$

of coherent sheaves on $C \times \mathbb{P}$ such that $\mathcal{V}$ is a vector bundle and $\mathcal{T}_0$ is a sheaf supported on $\{x\} \times \mathbb{P}$, which is a line bundle on $\mathbb{P}$. This means that $\mathbb{P}$ parametrises exact sequences

$$0 \longrightarrow W \longrightarrow V \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

of coherent sheaves on C such that $W \in \mathcal{SU}_C(r, \mathcal{O}(-x))$ and V is a vector bundle of rank r and of trivial determinant. Here Z denotes the reduced scheme supported on $x \in C$.

It is shown in [Ar-Sa, §5] that the space $\mathbb{P}$ is a fine moduli space of quasi-parabolic bundles and parametrises quasi-parabolic structures $V \longrightarrow \mathcal{O}_Z$ whose kernel is semi-stable. In particular there is a parabolic datum Δ which attaches weights $0 < \alpha_1 < \alpha_2 < 1$ (which are assumed to be small). Then we have

Theorem 3.1. *For every parabolic stable bundle $V \longrightarrow \mathcal{O}_Z$, the kernel W is semi-stable. Furthermore $\mathbb{P}$ is the moduli space of parabolic stable (which is same as stable) bundles $\mathcal{SU}_C(r, \mathcal{O}_C, \Delta)$ and the surjection $\mathcal{V} \longrightarrow \mathcal{T}_0$ is the universal family of parabolic bundles. Furthermore if $W \in \mathcal{SU}_C(r, \mathcal{O}(-x))$ then V is a semi-stable vector bundle.*

Proof. See [Ar-Sa, Theorem 5.0.3, Corollary 5.0.3]. □

In particular there is a *Hecke diagram* relating the moduli spaces $\mathcal{SU}_C(r, \mathcal{O}(-x))$ and $\mathcal{SU}_C(r, \mathcal{O}_C)$:

$$\begin{array}{ccc} \mathbb{P} & \stackrel{f}{\longrightarrow} & \mathcal{SU}_C(r, \mathcal{O}_C) \\ \downarrow \pi & & \\ \mathcal{SU}_C(r, \mathcal{O}(-x)). & & \end{array}$$

Furthermore, there is an open set $U := f^{-1}\mathcal{SU}^s_C(2, \mathcal{O}_C) \subset \mathbb{P}$ such that $f^{-1}V \simeq \mathbb{P}(V_x^*)$ for $V \in \mathcal{SU}^s_C(r, \mathcal{O}_C)$. In particular f restricts to a projection

(5) $$f_U : U \longrightarrow \mathcal{SU}^s_C(r, \mathcal{O}_C)$$

which is a $\mathbb{P}^{r-1}$-bundle, see [Ar-Sa, Remark 5.0.2]. The diagram

(6) $$\mathcal{SU}_C(r, \mathcal{O}(-x)) \stackrel{\pi}{\leftarrow} \mathbb{P} \supset U \stackrel{f_U}{\to} \mathcal{SU}^s_C(r, \mathcal{O}_C)$$

together with Hodge theory, projective bundle formulas and codimension estimates enabled Arapura-Sastry to compare the cohomologies of the two moduli spaces in (6) at least in low degrees. For our purpose, we recall the codimension estimate [Ar-Sa, p.17];

(7) $$\mathrm{codim}(\mathbb{P} - U) \geq 3$$

whenever $g \geq 3$. Altogether, the cohomology $H^3(\mathcal{SU}^s_C(r, \mathcal{O}_C), \mathbb{Q})$ has a pure Hodge structure of weight 3 and there is an isomorphism of pure Hodge structures [Ar-Sa, Theorem 8.3.1]:

(8) $$H^3(\mathcal{SU}_C(r, \mathcal{O}(-x)), \mathbb{Q}) \simeq H^3(\mathcal{SU}^s_C(r, \mathcal{O}_C), \mathbb{Q}).$$

Together with the isomorphism in Theorem 2.1, there is an isomorphism of Hodge structures:

(9) $$H^1(C, \mathbb{Q})(-1) \simeq H^3(\mathcal{SU}^s_C(r, \mathcal{O}_C), \mathbb{Q}).$$

We would like to extend this isomorphism to the cohomology $H^{2n-3}(\mathcal{SU}^s_C(r, \mathcal{O}_C), \mathbb{Q})$ as an isomorphism of Hodge structures, via the Lefschetz operator. Since $\mathcal{SU}^s_C(r, \mathcal{O}_C)$ is a non-compact smooth variety, the Hard Lefschetz theorem is not immediate. In general, this theorem does not hold for non-compact smooth varieties. Nevertheless, we investigate the action of the Lefschetz operator on the degree 3 rational cohomology group, which is relevant to our situation.

For this purpose, we look at the resolution $\mathcal{S}$ of $\mathcal{SU}_C(r,\mathcal{O}_C)$ constructed by Seshadri [Se2] together with an understanding of the exceptional loci of the resolution

$$g:\mathcal{S}\longrightarrow \mathcal{SU}_C(r,\mathcal{O}_C). \tag{10}$$

This map restricts to an isomorphism $g^{-1}\mathcal{SU}^s_C(r,\mathcal{O}_C)\simeq \mathcal{SU}^s_C(r,\mathcal{O}_C)$. The variety $\mathcal{S}$ is constructed as a moduli space of semi-stable vector bundles of rank r^2 and trivial determinant whose endomorphism algebra is a specialisation of the matrix algebra. This moduli space is a normal projective variety and it is proved to be a smooth variety only when the rank $r=2$. Hence we assume $r=2$ in the further discussion. See also other resolutions by Narasimhan-Ramanan [Na-Ra2] and Kirwan [Ki].

3.1. **Stratification of $\mathcal{S}$ when $r=2$.** We will recall the description of the exceptional loci of Seshadri's desingularisation given in [Ba, §3], [Ba-Se]. Recall that the singular locus of $\mathcal{SU}_C(2,\mathcal{O}_C)$ is parametrised by semi-stable bundles of the type $L\oplus L^{-1}$ for $L\in Jac(C)$. The inverse map i on $Jac(C)$ is given by $L\mapsto L^{-1}$. In other words, the Kummer variety $K(C):=\frac{Jac(C)}{<i>}$ is precisely the singular locus. Denote the image of the set of 2^{2g} fixed points by $K(C)_{fix}\subset K(C)$. There is a stratification

$$\mathcal{SU}_C(2,\mathcal{O}_C)=\mathcal{SU}^s_C(2,\mathcal{O}_C)\sqcup(K(C)-K(C)_{fix})\sqcup K(C)_{fix}. \tag{11}$$

The desingularisation $\mathcal{S}$ is stratified by the rank of a natural conic bundle on $\mathcal{S}$ [Ba, §3] and there is a filtration by closed subvarieties

$$\mathcal{S}=\mathcal{S}_0\supset\mathcal{S}_1\supset\mathcal{S}_2\supset\mathcal{S}_3 \tag{12}$$

such that $\mathcal{S}-\mathcal{S}_1=g^{-1}(\mathcal{SU}^s_C(2,\mathcal{O}_C))$ and $\mathcal{S}_{i+1}$ is the singular locus of $\mathcal{S}_i$.

The strata are described by the following:

Proposition 3.2. *1) The image $g(\mathcal{S}_1-\mathcal{S}_2)$ is precisely the middle stratum. In fact $\mathcal{S}_1-\mathcal{S}_2$ is a $\mathbb{P}^{g-2}\times\mathbb{P}^{g-2}$ bundle over $K(C)-K(C)_{fix}$.*

2) The image of $\mathcal{S}_2$ is precisely the deepest strata $K(C)_{fix}$ and $\mathcal{S}_2-\mathcal{S}_3$ is the disjoint union of 2^{2g} copies of a vector bundle of rank $g-2$ over the Grassmanian $G(2,g)$. The stratum $\mathcal{S}_3$ is the disjoint union of 2^{2g} copies of the Grassmanian $G(3,g)$.

Proof. See [Ba-Se, section 2]. □

We now note that the strata $\mathcal{S}_1,\mathcal{S}_2$ in the stratification $\mathcal{S}\supset\mathcal{S}_1\supset\mathcal{S}_2\supset\mathcal{S}_3$ are singular varieties. Moreover $\mathcal{S}_2\subset\mathcal{S}_1$ and $\mathcal{S}_3\subset\mathcal{S}_2$ are the singular loci respectively. Hence for our purpose, we look at the Borel-Moore homology theory (for example see [Pe-St, Chapter V, §6] for properties and the notion of weights) in the below discussion.

Before proving our main result, we recall the projective bundle formula of mixed Hodge structures:

Lemma 3.3. *Suppose $p : M \to N$ is a d-fold fibre product of $\mathbb{P}^{r-1}$-bundles (which need not be locally trivial in the Zariski topology) over a smooth quasi-projective variety N. Then, we have an equality of the mixed Hodge structures;*

$$W_iH^i(M,\mathbb{Q}) = \oplus_{j\geq 0}W_{i-2j}H^{i-2j}(N,\mathbb{Q}) \otimes H^{2j}((\mathbb{P}^{r-1})^d,\mathbb{Q}).$$

The dual statement gives the equality of the Borel-Moore homologies of M and N;

$$W_{-i}H_i(M,\mathbb{Q}) = \oplus_{j\geq 0}W_{2j-i}H_{i-2j}(N,\mathbb{Q}) \otimes H_{2j}((\mathbb{P}^{r-1})^d,\mathbb{Q}).$$

Proof. The proof is basically given in [Ar-Sa, Proposition 6.3.1]. Their proof is stated in terms of the 'full' cohomology $H^i(N,\mathbb{Q})$ and $H^i(M,\mathbb{Q})$, whereas for our purpose we restrict their formula to the bottom weight cohomology of M, which gives the formula as stated. Here we use the fact that the product of projective spaces have only pure Hodge structures and that the category of $\mathbb{Q}$-mixed Hodge structures is semi-simple. In fact the same proof also holds for the compactly supported cohomologies $H^i_c(M,\mathbb{Q})$ and $H^i_c(N,\mathbb{Q})$ and we can restrict it on the weight i-piece, to get the formula;

$$W_iH^i_c(M,\mathbb{Q}) = \oplus_{j\geq 0}W_{i-2j}H^{i-2j}_c(N,\mathbb{Q}) \otimes H^{2j}((\mathbb{P}^{r-1})^d,\mathbb{Q}).$$

The dual statement follows from the definition (see [Pe-St, Definition-Lemma 6.11]) of weights on the Borel-Moore homology of a variety R;

$$H_k(R,\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(H^k_c(R,\mathbb{Q}),\mathbb{Q}).$$

This has weights in the interval $[-k,0]$. In particular, we have the equality $W_{-k}H_k(R,\mathbb{Q}) := \mathrm{Hom}_{\mathbb{Q}}(W_kH^k_c(R,\mathbb{Q}),\mathbb{Q})$.

□

We also recall the following standard facts for the convenience of the reader:

Lemma 3.4. *Let X be a compact smooth variety of pure dimension n and $T \subset X$ be a closed subvariety.*

a) Then for all k we have a nonsingular pairing of mixed Hodge structures

$$H^k_T(X) \otimes H^{2n-k}(T) \to H^{2n}_T(X) \to \mathbb{Q}(-n).$$

Here $H^k_T(X)$ denotes the cohomology supported on T.

b) There is a long exact sequence

$$\to H^k_T(X) \to H^k(X) \to H^k(X-T) \to .$$

The group $H^k_T(X)$ has only weights $\geq k$ and the image of this group in $H^k(X)$ has weight k. In other words, the image is identified with the image of the group $W_kH^k_T(X)$.

c) (Poincaré duality) There is an isomorphism

$$H^{2n-k}_T(X) \simeq H_k(T)(-n)$$

of $\mathbb{Q}$-mixed Hodge structures.

Proof. a) See the discussion above [Pe-St, Corollary 6.14].

b) See [Pe-St, Corollary 6.14] and noting that $H^k(X)$ has a pure Hodge structure of weight k.

c) See [Ja, p.82, h) and p.92, Example 6.9].

□

Corollary 3.5. *The weight $(-i)$ piece $W_{-i}H_i(\mathcal{S}_1, \mathbb{Q})$ of the homology $H_i(\mathcal{S}_1, \mathbb{Q})$ of $\mathcal{S}_1$ is zero, whenever i is odd.*

Proof. We firstly note that the involution i acts as (-1) on the odd homology (respectively $(+1)$ on the even homology) of the Jacobian $J(C)$. The i-fixed homology classes precisely correspond to the homology classes on the quotient variety $K(C)$. This means that the homology of the Kummer variety $K(C)$ is zero in odd degrees. Look at the long exact homology sequence for the triple $(K(C), K(C) - K(C)_{fix}, K(C)_{fix})$ (see [Ja, p.81,§6, f)]):

$$\to H_i(K(C)_{fix}, \mathbb{Q}) \to H_i(K(C), \mathbb{Q}) \to H_i(K(C) - K(C)_{fix}, \mathbb{Q}) \to .$$

Since $K(C)_{fix}$ is a finite set of points, we conclude that the homology of the open variety $K(C) - K(C)_{fix}$ is also zero in odd degrees, except in degree $i = 1$. In this case, we obtain an injectivity $H_1(K(C) - K(C)_{fix}, \mathbb{Q}) \hookrightarrow H_0(K(C)_{fix}, \mathbb{Q})$. Since $H_0(K(C)_{fix}, \mathbb{Q})$ has weight 0, we conclude that $H_1(K(C) - K(C)_{fix}, \mathbb{Q})$ also has only weight 0. In particular, we have the vanishing $W_{-1}H_1(K(C) - K(C)_{fix}, \mathbb{Q}) = 0$.

Now we look at the homology of the triple $(\mathcal{S}_2, \mathcal{S}_2 - \mathcal{S}_3, \mathcal{S}_3)$: the long exact homology sequence for this triple is

$$\to H_i(\mathcal{S}_3, \mathbb{Q}) \to H_i(\mathcal{S}_2, \mathbb{Q}) \to H_i(\mathcal{S}_2 - \mathcal{S}_3, \mathbb{Q}) \to .$$

Note that the Grassmanian and vector bundles over Grassmanians have only algebraic homology, i.e., have homology only in even degrees. By Proposition 3.2 2), we know that $\mathcal{S}_2 - \mathcal{S}_3$ and $\mathcal{S}_3$ is made of such objects. Hence we conclude that $\mathcal{S}_2$ has all of its odd degree cohomology as zero.

Now look at the long exact homology sequence for the triple $(\mathcal{S}_1, \mathcal{S}_1 - \mathcal{S}_2, \mathcal{S}_2)$: here we note that the homology of the projective bundles or more generally flag varieties over a variety are generated by the homology of the base variety and the standard homology classes of powers of $\mathcal{O}(1)$ on these bundles. In particular these standard classes contribute only in the even homology of the total space. Since we have noticed above that $K(C) - K(C)_{fix}$ has only vanishing of the odd degree homology in degrees > 1, and in degree one case we have $W_{-1}H_1(K(C) - K(C)_{fix}, \mathbb{Q}) = 0$, it follows that $\mathcal{S}_1 - \mathcal{S}_2$ has in odd degree i, the vanishing of the bottom weight cohomology $W_{-i}H_i(\mathcal{S}_1 - \mathcal{S}_2, \mathbb{Q}) = 0$, by Lemma 3.3. We use the long exact homology sequence for the triple $(\mathcal{S}_1, \mathcal{S}_1 - \mathcal{S}_2, \mathcal{S}_2)$ and since $\mathcal{S}_2$ has vanishing odd degree homology we conclude that $\mathcal{S}_1$ has for an odd degree i, the bottom weight cohomology $W_{-i}H^i(\mathcal{S}_1, \mathbb{Q}) = 0$. □

3.2. **Hard Lefschetz isomorphism for $W_{odd}H^{odd}(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$.** Fix an ample class $\mathcal{L}$ on the moduli space $\mathcal{S}$ which restricts to an ample class $\mathcal{L}_s$ on $\mathcal{SU}^s_C(2,\mathcal{O}_C)$. Let $n := \dim\mathcal{S}$. Now we look at the long exact cohomology sequence for the triple $(\mathcal{S},\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathcal{S}_1)$ which is compatible with the Lefschetz operators:

$$\begin{array}{ccccc} \to H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q}) & \xrightarrow{g} H^{n-i}(\mathcal{S},\mathbb{Q}) & \xrightarrow{h} & H^{n-i}(\mathcal{SU}^s_C(r,\mathcal{O}_C),\mathbb{Q}) \to \\ & \downarrow \cup\mathcal{L}^i & & \downarrow \cup\mathcal{L}^i_s \\ \to H^{n+i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q}) & \xrightarrow{g'} H^{n+i}(\mathcal{S},\mathbb{Q}) & \xrightarrow{h'} & H^{n+i}(\mathcal{SU}^s_C(r,\mathcal{O}_C),\mathbb{Q}) \to . \end{array}$$

Here $H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ and $H^{n+i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ denotes the cohomology supported on $\mathcal{S}_1$.

Lemma 3.6. *The image of the cohomology $H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ (respectively $H^{n+i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$) under g (respectively g') is zero, whenever $n-i$ is odd. In particular the maps h,h' in the above long exact sequences are injective whenever $n-i$ is odd.*

Proof. Since $\mathcal{S}_1 \subset \mathcal{S}$ is a divisor, we can identify the group $H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ with the Borel–Moore homology $H_{n+i}(\mathcal{S}_1,\mathbb{Q}(n))$, which is actually the group $H_{n+i}(\mathcal{S}_1,\mathbb{Q})(-n)$, see Lemma 3.4 c). This group has weights $2(n)-(n+i)=n-i$ and higher, and the image of $H_{n+i}(\mathcal{S}_1,\mathbb{Q}(n))$ coincides with the image of $W_{n-i}(H_{n+i}(\mathcal{S}_1,\mathbb{Q})(-n))$, see Lemma 3.4 b).

Now we notice that since $\mathbb{Q}(-n)$ has weight $(2n)$, the group $W_{n-i}(H_{n+i}(\mathcal{S}_1,\mathbb{Q})(-n))$ is identified with the group $(W_{-n-i}H_{n+i}(\mathcal{S}_1,\mathbb{Q}))(-n)$. Since $n-i, n+i$ are odd, by Corollary 3.5, we know that $W_{-n-i}H_{n+i}(\mathcal{S}_1,\mathbb{Q})=0$. Hence we conclude that the image of the cohomology $H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ under g is zero. A similar computation shows that the group $H^{n+i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$ is identified with the group $W_{-n+i}H_{n-i}(\mathcal{S}_1,\mathbb{Q})(-n)$ and hence its image under g' is also zero. □

Since the images of h and h' correspond to the bottom weight cohomology of $\mathcal{SU}^s_C(2,\mathcal{O}_C)$ (see [De, p.39, Corollaire 3.2.17]), we can rewrite the above sequences as the following commutative diagram of exact sequences;

$$\begin{array}{ccccc} \to H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q}) & \xrightarrow{g} H^{n-i}(\mathcal{S},\mathbb{Q}) & \xrightarrow{h} & W_{n-i}H^{n-i}(\mathcal{SU}^s_C(r,\mathcal{O}_C),\mathbb{Q}) \to 0 \\ \downarrow \cup\mathcal{L}^i_{\mathcal{S}_1} & \downarrow \cup\mathcal{L}^i & & \downarrow \cup(\mathcal{L}^i_s)_W \\ \to H^{n+i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q}) & \xrightarrow{g'} H^{n+i}(\mathcal{S},\mathbb{Q}) & \xrightarrow{h'} & W_{n+i}H^{n+i}(\mathcal{SU}^s_C(r,\mathcal{O}_C),\mathbb{Q}) \to 0. \end{array}$$

Lemma 3.7. *The map $\cup(\mathcal{L}^i_s)_W$ between the bottom weight cohomologies in the above long exact sequences is an isomorphism, whenever $n-i$ is odd, as a morphism of Hodge structures.*

Proof. Since the line bundle $\mathcal{L}$ restricts to an ample class on $\mathcal{SU}^s_C(2,\mathcal{O}_C)$, the Lefschetz operator $\cup\mathcal{L}^i$ induces an operator on the cohomology $H^{n-i}_{\mathcal{S}_1}(\mathcal{S},\mathbb{Q})$. Applying the Hard Lefschetz isomorphism on $\mathcal{S}$, we have the isomorphism of the operator $\cup\mathcal{L}^i$, as morphisms of Hodge structures. By Lemma 3.6, we deduce that the morphisms h and h' are injective and onto the bottom weight cohomologies. Since $\cup\mathcal{L}^i$ is an isomorphism we obtain that $\cup(\mathcal{L}^i_s)_W$ is also an isomorphism. □

Lemma 3.8. *The Lefschetz operator*

$$\cup\mathcal{L}_s^{n-3} : H^3(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q}) \to H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$$

is injective as a morphism of Hodge structures.

Proof. Consider the above long exact sequences when $n-i=3, n+i=2n-3$. Using Lemma 3.7, we know that the maps h, h' in the above long exact cohomology sequence on the cohomology are injective. Since the Hard Lefschetz isomorphism holds on $\mathcal{S}$, the operator $\cup\mathcal{L}^{n-3}$ is an isomorphism. Furthermore, by [Ar-Sa, Theorem 8.3.1], the cohomology $H^3(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$ has a pure Hodge structure and hence the map h is an isomorphism onto $H^3(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$. This implies that $\cup\mathcal{L}_s^{n-3}$ is injective. Moreover, the maps in the above long exact sequence are morphisms of mixed Hodge structures and since $\cup\mathcal{L}^{n-3}$ is an isomorphism of Hodge structure, we obtain the last assertion.

□

Corollary 3.9. *There is an injectivity*

$$H^1(C,\mathbb{Q}) \hookrightarrow H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$$

of Hodge structures. In particular the intermediate Jacobian

$$IJ_1(W_{2n-3})\otimes\mathbb{Q} := \frac{W_{2n-3}H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C,\mathbb{C})}{F^{n-1}+W_{2n-3}H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})}$$

is isomorphic to the Jacobian $Jac(C)\otimes\mathbb{Q}$.

Proof. Use (9) together with Lemma 3.8 to conclude that $H^1(C,\mathbb{Q})$ is a sub-Hodge structure of the mixed Hodge structure on $H^{2n-3}(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q})$. Using Lemma 3.7, we deduce the last assertion. □

3.3. **Surjectivity of the Abel–Jacobi map AJ_1^s onto $Jac(C)$.** Consider the Abel–Jacobi map

$$AJ_1^s : CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \xrightarrow{AJ_1^s} IJ_1(W_{2n-3})\otimes\mathbb{Q} \simeq Jac(C)\otimes\mathbb{Q}.$$

We will prove that the map AJ_1^s is surjective.

Recall the Hecke diagram from the previous subsection:

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{f} & \mathcal{SU}_C(2,\mathcal{O}_C) \\ \downarrow\pi & & \\ \mathcal{SU}_C(2,\mathcal{O}(-x)). & & \end{array}$$

Denote the intermediate Jacobians of a smooth projective variety Y of dimension l by

$$IJ^2(Y) := \frac{H^3(Y,\mathbb{C})}{F^2+H^3(Y,\mathbb{Z})},\ IJ_1(Y) := \frac{H^{2l-3}(Y,\mathbb{C})}{F^l+H^{2l-3}(Y,\mathbb{Z})}.$$

We will use these notations for the spaces $\mathcal{SU}_C(2,\mathcal{O}(-x))$ and the $\mathbb{P}^1$-bundle $\mathbb{P}$.

Lemma 3.10. *a) We have the following isomorphisms of the intermediate Jacobians*

$$\begin{aligned} IJ_1(\mathbb{P}) &\simeq IJ_1(\mathcal{SU}_C(2,\mathcal{O}(-x))) \\ IJ^2(\mathcal{SU}_C(2,\mathcal{O}(-x))) &\simeq Jac(C)\otimes\mathbb{Q}. \end{aligned}$$

b) Similar isomorphisms hold for the Chow groups:

$$\begin{aligned} CH_1(\mathbb{P})_{hom} &= \alpha.\pi^*CH_1(\mathcal{SU}_C(2,\mathcal{O}(-x)))_{hom}, \\ CH^2(\mathbb{P})_{hom} &= CH^2(\mathcal{SU}_C(2,\mathcal{O}(-x)))_{hom}, \\ CH^2(U)_{hom}\otimes\mathbb{Q} &\stackrel{\simeq}{\leftarrow} CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}. \end{aligned}$$

Here $\alpha := c_1(\mathcal{O}_{\mathbb{P}}(1))$.

Proof. a) There is a decomposition

$$H^{2n-1}(\mathbb{P},\mathbb{Q}) = \pi^*H^{2n-1}(\mathcal{SU}_C(2,\mathcal{O}(-x)),\mathbb{Q}) \oplus \alpha.\pi^*H^{2n-3}(\mathcal{SU}_C(2,\mathcal{O}(-x)),\mathbb{Q}).$$

The first cohomology piece is zero since it is dual to $H^1(\mathcal{SU}_C(2,\mathcal{O}(-x)),\mathbb{Q})$ which is zero. This implies that there is an isomorphism of the intermediate Jacobians

$$IJ_1(\mathcal{SU}_C(2,\mathcal{O}(-x)) \simeq IJ_1(\mathbb{P}).$$

Again, by Hard Lefschetz isomorphism, we have the isomorphism

$$IJ^2(\mathcal{SU}_C(2,\mathcal{O}(-x))) \simeq Jac(C)\otimes\mathbb{Q}.$$

b) Using the projective bundle formula [Fu, p.64], similar statements hold on the Chow groups of cycles homologous to zero, as asserted. Indeed, we note that

$$\begin{aligned} CH^2(\mathbb{P})\otimes\mathbb{Q} &= CH_{n-1}(\mathbb{P})\otimes\mathbb{Q} \\ &= \pi^*CH_{n-2}(\mathcal{SU}_C(2,\mathcal{O}(-x)))\otimes\mathbb{Q} + \alpha.\pi^*CH_{n-1}(\mathcal{SU}_C(2,\mathcal{O}(-x)))\otimes\mathbb{Q}. \end{aligned}$$

Since $CH_{n-1}(\mathcal{SU}_C(2,\mathcal{O}(-x)))\otimes\mathbb{Q} = \mathbb{Q}$ ([Ra, Proposition 3.4]), we conclude the equality

$$CH^2(\mathbb{P})_{hom}\otimes\mathbb{Q} = CH^2(\mathcal{SU}_C(2,\mathcal{O}(-x)))_{hom}\otimes\mathbb{Q}.$$

Also, since $CH_0(\mathcal{SU}_C(2,\mathcal{O}(-x)))\otimes\mathbb{Q} = \mathbb{Q}$, the projective bundle formula gives the equality

$$CH_1(\mathbb{P})_{hom}\otimes\mathbb{Q} = \alpha.\pi^*CH_1(\mathcal{SU}_C(2,\mathcal{O}(-x)))_{hom}\otimes\mathbb{Q}.$$

Similarly, since $\mathrm{Pic}(\mathcal{SU}^s_C(2,\mathcal{O}_C))\otimes\mathbb{Q} = \mathbb{Q}$, we conclude the isomorphism

$$CH^2(U)_{hom}\otimes\mathbb{Q} \stackrel{\simeq}{\leftarrow} CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}.$$

□

Lemma 3.11. *There is a commutative diagram*

$$\begin{array}{ccc} CH^2(\mathcal{SU}_C(2,\mathcal{O}(-x)))_{hom}\otimes\mathbb{Q} & \stackrel{\simeq}{\longrightarrow} & CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \\ \downarrow AJ^2 & & \downarrow AJ^2_s \\ Jac(C)\otimes\mathbb{Q} & \stackrel{\simeq}{\longrightarrow} & Jac(C)\otimes\mathbb{Q}. \end{array}$$

Furthermore, the maps AJ^2 and AJ^2_s are isomorphisms, whenever $g \geq 3$.

Proof. Firstly, by Lemma 3.10, we have an isomorphism $IJ^2(\mathcal{SU}_C(2,\mathcal{O}(-x))) \simeq Jac(C)\otimes \mathbb{Q}$ and by (9) there is an isomorphism $Jac(C)\otimes\mathbb{Q} \simeq IJ^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))\otimes\mathbb{Q}$. Hence the maps AJ^2 and AJ^2_s are just the Abel-Jacobi maps. Consider the following maps on the cohomology

$$H^3(\mathcal{SU}_C(2,\mathcal{O}(-x)),\mathbb{Q}) \xrightarrow{\pi^*} H^3(\mathbb{P},\mathbb{Q}) \xrightarrow{s^0} H^3(U,\mathbb{Q}) \xleftarrow{f^*_U} H^3(\mathcal{SU}^s_C(2,\mathcal{O}_C),\mathbb{Q}). \tag{13}$$

Then all the above maps are isomorphisms, by [Ar-Sa, Theorem 8.3.1]. These are also isomorphisms of pure Hodge structures and hence induce isomorphisms of the associated intermediate Jacobians

$$IJ^2(\mathcal{SU}_C(2,\mathcal{O}(-x)) \simeq IJ^2(\mathbb{P}) \simeq IJ^2(U) \simeq IJ^2(\mathcal{SU}^s_C(2,\mathcal{O}_C)). \tag{14}$$

Similarly there are maps between the rational Chow groups

$$CH^2(\mathcal{SU}_C(2,\mathcal{O}(-x))_{hom}\otimes\mathbb{Q} \xrightarrow{\pi^*} CH^2(\mathbb{P})_{hom}\otimes\mathbb{Q} \xrightarrow{t^0} CH^2(U)_{hom}\otimes\mathbb{Q} \xleftarrow{f^*_U} CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}. \tag{15}$$

Using Lemma 3.10, we conclude that π^* and f^*_U are isomorphisms. Also, there is a localization exact sequence (see [Fu, Proposition 1.8, p.21])

$$\to CH_{n-1}(\mathbb{P}-U)\otimes\mathbb{Q} \to CH_{n-1}(\mathbb{P})\otimes\mathbb{Q} \xrightarrow{t^0} CH_{n-1}(U)\otimes\mathbb{Q} \to 0.$$

Using the codimension estimate (7), we conclude that $CH_{n-1}(\mathbb{P}-U)\otimes\mathbb{Q} = 0$, i.e., t^0 is an isomorphism, whenever $g \geq 3$. This implies that all the maps in (15) are isomorphisms and compatible with the Abel–Jacobi maps into the objects in (14). This gives the commutative diagram as in the statement of the lemma. By [Ba-Kg-Ne, p.10], we know that AJ^2 is an isomorphism. This implies that AJ^2_s is also an isomorphism.

□

Lemma 3.12. *There is a commutative diagram*

$$\begin{array}{ccc} CH^2(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} & \stackrel{\mathcal{L}^{n-3}}{\rightarrow} & CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \\ \downarrow AJ^2_s & & \downarrow AJ^s_1 \\ Jac(C)\otimes\mathbb{Q} & \stackrel{\mathcal{L}^{n-3}_s}{\rightarrow} & IJ_1(W_{2n-3})\otimes\mathbb{Q}. \end{array}$$

such that $\mathcal{L}^{n-3}_s$ maps isomorphically onto $Jac(C)\otimes\mathbb{Q} \simeq IJ_1(W_{2n-3})\otimes\mathbb{Q}$.

Proof. We just need to note that the Lefschetz operator acts compatibly on the Chow groups and on the cohomology as a morphism of Hodge structures. This gives the commutative diagram. Also, by Lemma 3.8 and Corollary 3.9, the map $\mathcal{L}^{n-3}_s$ maps isomorphically onto $Jac(C)\otimes\mathbb{Q}$. □

Consider the Abel–Jacobi map

$$AJ^s_1 : CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q} \xrightarrow{AJ^s_1} IJ_1(W_{2n-3})\otimes\mathbb{Q} \simeq Jac(C)\otimes\mathbb{Q}.$$

Corollary 3.13. *The Abel-Jacobi map AJ^s_1 is surjective.*

Proof. We use the commutative diagram in Lemma 3.12 and the isomorphism of $\mathcal{L}_s^{n-3}$ onto $Jac(C) \otimes \mathbb{Q}$. By Lemma 3.11, we know that the map AJ_s^2 is an isomorphism. Hence we conclude that the map AJ_1^s is surjective. □

4. Chow generation of one cycles on $\mathcal{SU}_C(2, \mathcal{O}_C)$

A study of Fano manifolds with Picard number one with respect to the geometry of the variety of tangent directions to the minimal rational curves, has been studied by J-M. Hwang and N. Mok in a series of papers (see [Hw-Mo] for a survey). The moduli space $\mathcal{SU}_C^s(2, \mathcal{O}_C)$ is a Fano manifold with Picard number one and the minimal rational curves are the 'Hecke curves' introduced by Ramanan and Narasimhan [Na-Ra2]. Suppose $\mathcal{L}$ is the ample generator of $\mathcal{SU}_C(2, \mathcal{O}_C)$ then the dualizing class K is equal to $-4\mathcal{L}$ ([Be]). Furthermore, a Hecke curve has degree 4 with respect to $-K$. Hence it has degree one with respect to $\mathcal{L}$ (see also [Hw]). In this section, we include the results of J-M. Hwang which shows that the Hecke curves generate the Chow group of one cycles on the moduli space $\mathcal{SU}_C(2, \mathcal{O}_C)$.

With notations as in the previous section or [Ar-Sa], there is a fibration

$$\begin{array}{ll} \mathcal{P} & \xrightarrow{f} \mathcal{SU}_C(r, \mathcal{O}_C) \\ \downarrow\pi & \\ \mathcal{S} & \\ \downarrow\psi & \\ C & \end{array}$$

such that the fibre of the morphism ψ at a point $x \in C$ is the moduli space $\mathcal{SU}_C(r, \mathcal{O}(-x))$. The variety $\mathcal{P}$ is a $\mathbb{P}^{r-1}$-bundle over $\mathcal{S}$ and restricting over a fibre $\psi^{-1}(x)$ gives precisely the Hecke correspondence used in the previous section. The image under f of the lines in the fibres of the projection π are the Hecke curves on $\mathcal{SU}_C(r, \mathcal{O}_C)$.

Firstly, we look at a variant of a theorem of Kollar [Ko, Proposition 3.13.3], on the Chow generation of one cycles on a variety, due to H-M. Hwang:

Proposition 4.1. *(Hwang) Let X be a normal projective variety. Let $\mathcal{M} \subset \mathrm{Chow}(X)$ be a closed subscheme of the Chow scheme of X such that all members of $\mathcal{M}$ are irreducible and reduced curves on X. For a general point $x \in X$, let $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ be the closure of the union of the tangent vectors to members of $\mathcal{M}$ passing through x, which are smooth at x. Suppose that $\mathcal{C}_x$ is non-degenerate in $\mathbb{P}T_x(X)$. Then the Chow group of 1-cycles $\mathrm{CH}_1(X) \otimes \mathbb{Q}$ is generated by members of $\mathcal{M}$.*

Proof. Applying Theorem IV.4.13 of [Ko], we have an open subvariety $X^o \subset X$ and a morphism $\pi : X^o \to Z^o$ with connected fibers such that for every $z \in Z^o$, any two general points of $\pi^{-1}(z)$ can be connected by a connected chain consisting of members of $\mathcal{M}$ of length at most $\dim X$ and all members of $\mathcal{M}$ through a point $x \in \pi^{-1}(z)$ are contained in the closure of $\pi^{-1}(z)$. By the assumption on the non-degeneracy of $\mathcal{C}_x$ in $\mathbb{P}T_x(X)$, this implies that Z^o is a point. It follows that any two general points of X can be connected chain consisting of members of $\mathcal{M}$ of length at most $\dim X$. Let $Y \subset \mathrm{Chow}(X)$ be the closed subscheme parametrizing connected 1-cycles each component of which is a member of $\mathcal{M}$. In the notation of Lemma IV.3.4 of [Ko], $u^{(2)} : U \times_Y U \to X \times X$ is dominant, where $g : U \to Y$ is the universal family and $u : U \to X$ is the cycle map. Since U is complete, $u^{(2)}$ is surjective as in the proof of Corollary IV.3.5 of [Ko]. Thus we can apply Proposition 3.13.3 of [Ko] to conclude that $\mathrm{CH}_1(X) \otimes \mathbb{Q}$ is generated by members of $\mathcal{M}$. □

This proposition can now be used to obtain the Chow generation of one cycles on the moduli space $\mathcal{SU}_C(r, \mathcal{O}_C)$.

Corollary 4.2. *(Hwang) Let $X = \mathcal{SU}_C(r, \mathcal{O}_C)$ for a curve C of genus ≥ 4. Then $\mathrm{CH}_1(X) \otimes \mathbb{Q}$ is generated by Hecke curves. Moreover, there is a surjection*

$$CH_0(C) \otimes \mathbb{Q} \longrightarrow CH_1(\mathcal{SU}_C(r, \mathcal{O}_C)) \otimes \mathbb{Q}.$$

Proof. By Theorem 3 of [Hw2], the union of the tangent vectors to Hecke curves through a general point $y \in X$ is a non-degenerate subvariety $\mathcal{C}_y \subset \mathbb{P}T_y(X)$. Since Hecke curves have degree one with respect to the generator of $\mathrm{Pic}(X)$, they form a closed subscheme $\mathcal{M}$ in $\mathrm{Chow}(X)$. Thus Proposition 4.1 applies. This implies that there is a surjection

$$CH_0(\mathcal{S}) \otimes \mathbb{Q} \to CH_1(\mathcal{SU}_C(r, \mathcal{O}_C)) \otimes \mathbb{Q}.$$

For any $x \in C$, the moduli space $\mathcal{SU}_C(r, \mathcal{O}(-x))$ is rationally connected and we have the triviality $CH_0(\mathcal{SU}_C(r, \mathcal{O}(-x)) \otimes \mathbb{Q} = \mathbb{Q}$. This gives us an isomorphism

$$CH_0(C) \otimes \mathbb{Q} \simeq CH_0(\mathcal{S}) \otimes \mathbb{Q}.$$

Altogether, we now get a surjection

$$CH_0(C) \otimes \mathbb{Q} \longrightarrow CH_1(\mathcal{SU}_C(r, \mathcal{O}_C)) \otimes \mathbb{Q}.$$

□

Corollary 4.3. *The Abel-Jacobi map*

$$AJ_1^s : CH_1(\mathcal{SU}^s_C(2, \mathcal{O}_C))_{hom} \otimes \mathbb{Q} \longrightarrow Jac(C) \otimes \mathbb{Q}$$

is an isomorphism, whenever $g \geq 4$. This extends to an isomorphism

$$CH_1(\mathcal{SU}_C(2, \mathcal{O}_C))_{hom} \otimes \mathbb{Q} \longrightarrow Jac(C) \otimes \mathbb{Q}.$$

Proof. By Corollary 4.2, there is a surjection

$$Jac(C) \otimes \mathbb{Q} \longrightarrow CH_1(\mathcal{SU}_C(r, \mathcal{O}_C))_{hom} \otimes \mathbb{Q}.$$

Furthermore, by Corollary 3.13, the Abel-Jacobi map AJ_1^s

$$CH_1(\mathcal{SU}^s_C(2, \mathcal{O}_C))_{hom} \otimes \mathbb{Q} \to Jac(C) \otimes \mathbb{Q}$$

is surjective and the localization map

$$CH_1(\mathcal{SU}_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}\to CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}$$

is also surjective [Fu, p.21]. Putting these facts together, we note that all the maps in the following sequence

$$Jac(C)\otimes\mathbb{Q}\to CH_1(\mathcal{SU}_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}\to CH_1(\mathcal{SU}^s_C(2,\mathcal{O}_C))_{hom}\otimes\mathbb{Q}\to Jac(C)\otimes\mathbb{Q} \tag{16}$$

are surjective. Since the composed map $Jac(C)\otimes\mathbb{Q}\to Jac(C)\otimes\mathbb{Q}$ is surjective, it is an isomorphism. Hence all the maps in (16) are isomorphisms.

□

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