Order and chaos in wave propagation

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A physical phenomenon : harmonics (stationary vibration modes).

A mathematical theorem : the spectral decomposition of the laplacian.

The wave equation :

$$u(t, x, y)$$

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$$\frac{\partial^2}{\partial t^2} u = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u$$

$$\frac{\partial^2}{\partial t^2} u = c^2 \Delta u$$

An abstract language – allowing unified treatment of all wave phenomena, regardless of their physical origin (sound, electromagnetic waves, seismic waves,...)

The wave equation has special "stationary" solutions, i.e. of the form

$$u(t, x, y) = e^{i\omega t}\psi(x, y),$$

also called "eigenmodes", "characteristic modes".
The function ψ must satisfy $\Delta\psi = -\frac{\omega^2}{c^2}\psi$.

FIGURE: Ernst Chladni (1756-1827)

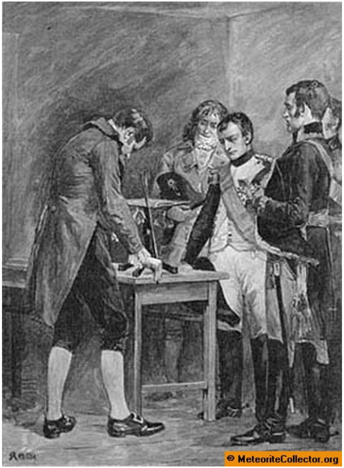
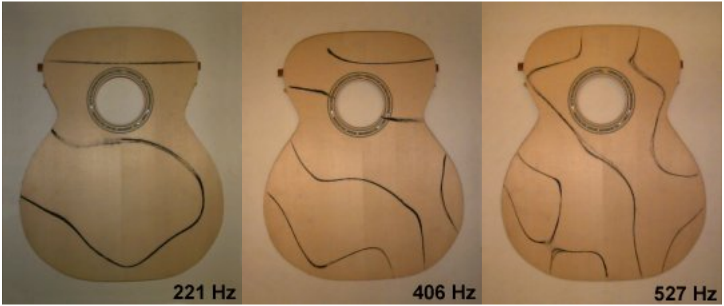


FIGURE: Sophie Germain (1776-1831)

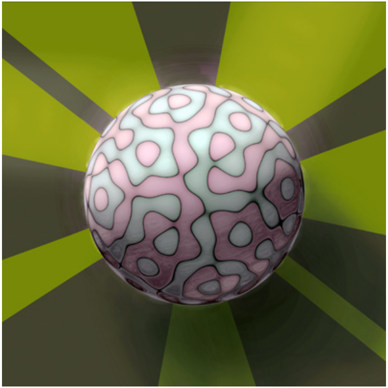
Doing research in math ?
Observing the world... answering questions... always leads to new questions !

Can we calculate explicitly the eigenfrequencies and eigenmodes ?
it depends on the shape of the membrane, but in general the answer is NO.

Nodal lines for a guitar table :



Nodal lines on a sphere :



[Eric Heller Gallery]

(R. Courant's theorem) For the *n*-th eigenmode, the nodal lines cut the membrane into at most *n* pieces (= nodal domains).

(Pleijel's theorem) In dimension ≥ 2. For *n* large enough, the nodal lines of the *n*-th eigenmode cut the membrane into at most αn pieces with $\alpha = 0,54$.

The nodal lines seem to “invade” the domain, to form a denser and denser family of curves, as the frequency increases.
Mathematical proof ?

(Donnelly-Fefferman 1988) The total length of nodal lines grows proportionnally to frequency :

$$Z_\psi = \{x, \psi(x) = 0\}.$$

$$C_1\omega_n \leq length(Z_{\psi_n}) \leq C_2\omega_n.$$

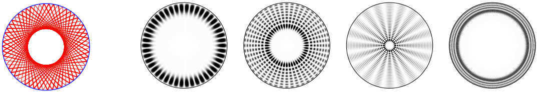
(proven if the boundary is analytic. Conjectured by Yau to hold always).

(Egorov-Kondratiev 1996, Nazarov-Polterovich-Sodin 2005) The "spacing" between nodal lines decreases like the inverse of frequency.
Let r_{ψ_n} be the "inner radius" (radius of a ball included in a nodal domain). Then

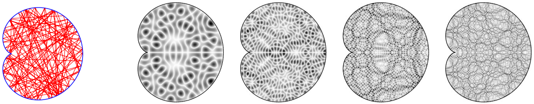
$$\frac{C_1}{\omega_n} \leq r_{\psi_n} \leq \frac{C_2}{\omega_n}$$

$n = 100$ $n = 1000$ $n = 1500$ $n = 2000$

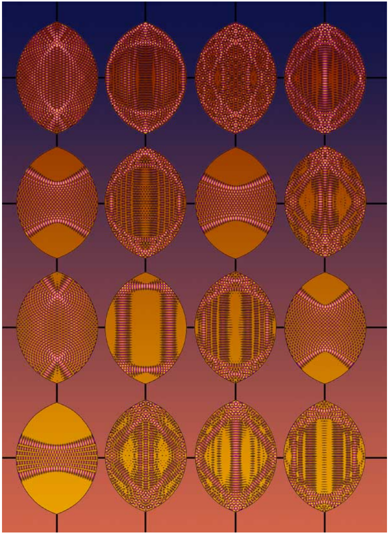
Billard régulier



Billard chaotique



How to explain the following patterns ?



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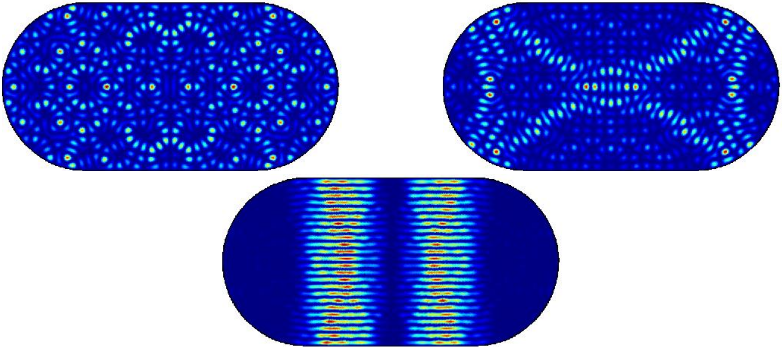

[Pär Kurlberg]

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and more generally, for $a \in C^0(\Omega)$,

$$\sum_{n, \omega_n \leq \lambda} \int_\Omega a(x,y)|\psi_n(x,y)|^2 dx\, dy \sim_{\lambda \longrightarrow +\infty} \frac{1}{4\pi} \lambda^2 \int_\Omega a(x,y) dx\, dy.$$

(Here (ψ_n) is an orthonormal basis of $L^2(\Omega)$ such that $\Delta \psi_n = -\omega_n^2 \psi_n$)

$$\frac{1}{N(\lambda)} \sum_{n,\omega_n \leq \lambda} \int a(x,y)|\psi_n(x,y)|^2 dx\, dy$$

$$\underset{\lambda \longrightarrow +\infty}{\longrightarrow} \frac{1}{\mathrm{Area}(\Omega)} \int_\Omega a(x,y) dx\, dy$$

$$\frac{1}{N(\lambda)} \sum_{n, \omega_n \leq \lambda} \int a(x,y) |\psi_n(x,y)|^2 dx\, dy$$

$$\underset{\lambda \longrightarrow +\infty}{\longrightarrow} \frac{1}{\mathrm{Area}(\Omega)} \int_\Omega a(x,y) dx\, dy$$

Individual behaviour of $|\psi_n(x,y)|^2$ as $n \longrightarrow +\infty$?

Do we have

$$\int a(x,y)|\psi_n(x,y)|^2 dx\, dy \underset{\lambda \longrightarrow +\infty}{\longrightarrow} \frac{1}{\mathrm{Area}(\Omega)} \int_\Omega a(x,y) dx\, dy?$$

If this is true, this means that the eigenfunctions are becoming uniformly spread out for high frequencies, and is interpreted as their having a "disordered" behaviour.

The answer seems to depend on the geometry of the domain.

For "negatively curved manifolds", it is believed that

$$\int a(x,y)|\psi_n(x,y)|^2 dx\, dy \underset{\lambda \longrightarrow +\infty}{\longrightarrow} \frac{1}{\mathrm{Area}(\Omega)} \int_\Omega a(x,y) dx\, dy.$$

Quantum Unique Ergodicity conjecture 1992, proven in some cases by E. Lindenstrauss (2002)

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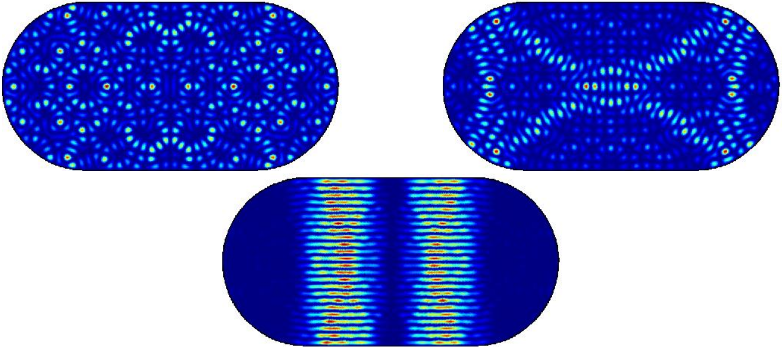
For the stadium billiard, Hassell has proved in 2008 that QUE is false : there are families of eigenfunctions that concentrate inside the rectangle.



[Pär Kurlberg]

"Can one hear the shape of a drum ?"

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

- If one hears the harmonics produced by a drum, can one guess its shape ?
- Is it possible for two membranes with different shapes to produce the same harmonics ?

One can prove mathematically that :

- if two membranes have the same harmonics, they must have the same area ; this comes from the Weyl law :

$$|\{n, \omega_n \leq \lambda\}|| \sim_{\lambda \longrightarrow +\infty} \frac{1}{4\pi}\mathrm{Area}(\Omega)\lambda^2.$$

- if two membranes have the same harmonics, they must have the same perimeter ;
- if a membrane has the same harmonics as a circular membrane, it must be circular ;
- if two rectangular membranes have the same harmonics, the rectangles must be the same.

BUT...

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, July 1992

ONE CANNOT HEAR THE SHAPE OF A DRUM

CAROLYN GORDON, DAVID L. WEBB, AND SCOTT WOLPERT

ABSTRACT. We use an extension of Sunada's theorem to construct a nonisometric pair of isospectral simply connected domains in the Euclidean plane, thus answering negatively Kac's question, "can one hear the shape of a drum?" In order to construct simply connected examples, we exploit the observation that an orbifold whose underlying space is a simply connected manifold with boundary need not be simply connected as an orbifold.

ONE CANNOT HEAR THE SHAPE OF A DRUM

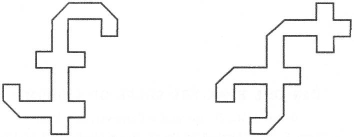


FIGURE 1