# CHOW–KÜNNETH DECOMPOSITION FOR A RATIONAL HOMOGENEOUS BUNDLE OVER A VARIETY

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Abstract. In this paper, we investigate the existence of a Chow-Künneth decomposition for a rational homogeneous bundle $Z \to S$ over a smooth variety. Absolute Chow-Künneth projectors are exhibited for $Z$ whenever $S$ has a Chow–Künneth decomposition and $Z$ is locally trivial over an étale atlas of $S$.

## 1. Introduction

Suppose $X$ is a nonsingular projective variety of dimension $n$ defined over the complex numbers. Let $CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension $i$ algebraic cycles modulo rational equivalence. Jacob Murre [Mu2], [Mu3] has made the following conjecture which leads to a filtration on the rational Chow groups:

**Conjecture**: The motive $h(X) := (X, \Delta_X)$ of $X$ has a Chow-Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in CH^n(X \times X) \otimes \mathbb{Q}$$

such that $\pi_i$ are orthogonal projectors (see §2.2).

Some examples where this conjecture is verified are: curves, surfaces, a product of a curve and surface [Mu], [Mu3], abelian varieties and abelian schemes [Sh],[De-Mu], uniruled threefolds [dA-Ml], elliptic modular varieties [Go-Mu], [GHMu2]), universal families over Picard modular surfaces [MWYK] and finite group quotients (maybe singular) of abelian varieties [Ak-Jo], some varieties with a nef tangent bundles [Iy], open moduli spaces of smooth curves [Iy-Ml], universal families over some Shimura surfaces [Mi].

In [Iy], we considered varieties which have a nef tangent bundle and obtained explicit Chow–Künneth projectors in small dimensions. Using the structure theorems of Campana and Peternell [Ca-Pe] and Demailly-Peternell-Schneider [DPS], we know that such a variety $X$ admits a finite étale surjective cover $X' \to X$ such that $X' \to A$ is a bundle of smooth Fano varieties over an abelian variety. Furthermore, any fibre which is a smooth Fano variety necessarily has a nef tangent bundle. It is an open question [Ca-Pe, p.170] whether such a Fano variety is a rational homogeneous variety. We showed in [Iy] that whenever the étale cover is a relative cellular variety over $A$ or if it admits a relative Chow–Künneth decomposition, then $X'$ and $X$ have a Chow–Künneth decomposition.

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In this paper, we weaken the hypothesis on the cover $X' \to X$ as above and obtain a Chow–Künneth decomposition whenever $X' \to A$ is a rational homogeneous bundle, which is locally trivial, over an étale atlas of the abelian variety $A$. This strengthens the results in [Iy] and we obtain a Chow–Künneth decomposition for a larger class of varieties which have a nef tangent bundle.

We state the result and proofs, in a more general situation.

**Theorem 1.1.** *Suppose $S$ is a smooth projective variety over the complex numbers. Let $G$ be a connected reductive algebraic group and let $Z$ be a rational $G$ homogeneous space over the variety $S$. Assume that $S$ has a Chow-Künneth decomposition and $Z \to S$ has local trivializations over étale open subsets of $S$ and which cover $S$. Then the following hold:*

*a) the motive of $Z$ has an absolute Chow–Künneth decomposition.*

*b) the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of $S$ with the twisted Tate motive.*

See also results on Chow groups of relative cellular spaces by B. Koeck [Ko] and by Nenashev-Zainoulline [Ne-Za].

The main observation in the proof is to note that a rational homogeneous bundle as above is étale locally a relative cellular variety. Hence we can construct relative Chow–Künneth projectors (in the sense of [De-Mu]) over étale coverings of $S$. These projectors lie in the subspace generated by the relative algebraic cells. The corresponding relative cohomology classes patch up since they lie in the subspace generated by the relative analytic cells. Hence the relative orthogonal projectors can be patched up as algebraic cycles to obtain relative projectors, in the rational Chow groups of the associated regular stack [Gi]. In this case, we show that the relative Chow–Künneth projectors over the regular stack descend to relative Chow–Künneth projectors for $Z \to S$ (see Corollary 3.6). The criterion of Gordon-Hanamura-Murre [GHMu2], for obtaining absolute Chow–Künneth projectors from relative Chow–Künneth projectors can be directly applied, see Proposition 3.7.

It is of interest to obtain the conclusions of the above theorem, by removing the assumption on étale local triviality, which we have been unable to treat in this paper. A positive answer will give us a fair understanding of the motive of a smooth projective variety with a nef tangent bundle.

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## 2. Preliminaries

We work over the field of complex numbers in this paper. We begin by recalling the standard constructions of the category of motives. Since this is fairly discussed in the literature, we give a brief account and refer to [Mu2], [Sc] for details.

2.1. **Category of motives.** The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. For an object $X$ of $\mathcal{V}$, let $CH^i(X)_{\mathbb{Q}} := CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension $i$ algebraic cycles modulo rational equivalence. Suppose $X, Y \in Ob(\mathcal{V})$ and $X = \cup X_i$ be a decomposition into connected components $X_i$ and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X, Y) = \oplus_i CH^{d_i+r}(X_i \times Y)_{\mathbb{Q}}$ is the group of correspondences of degree $r$ from $X$ to $Y$.

We will use the standard framework of the category of Chow motives $\mathcal{M}_{rat}$ in this paper and refer to [Mu2] for details. We denote the category of motives $\mathcal{M}_{\sim}$, where $\sim$ is any equivalence, for instance $\sim$ is homological or numerical equivalence. When $S$ is a smooth variety, we also consider the category of relative Chow motives $CH\mathcal{M}(S)$ which was introduced in [De-Mu] and [GHMu]. When $S =$ Spec $\mathbb{C}$ then the category $CH\mathcal{M}(S) = \mathcal{M}_{rat}$.

2.2. **Chow–Künneth decomposition for a variety.** Suppose $X$ is a nonsingular projective variety over $\mathbb{C}$ of dimension $n$. Let $\Delta_X \subset X \times X$ be the diagonal. Consider the Künneth decomposition of the class of $\Delta$ in the Betti Cohomology:

$$\Delta_X = \oplus_{i=0}^{2n} \pi_i^{hom}$$

where $\pi_i^{hom} \in H^{2n-i}(X) \otimes H^i(X)$.

**Definition 2.1.** *The motive of $X$ is said to have Künneth decomposition if each of the classes $\pi_i^{hom}$ are algebraic and are projectors, i.e., $\pi_i^{hom}$ is the image of an algebraic cycle $\pi_i$ under the cycle class map from the rational Chow groups to the Betti cohomology and satisfying $\pi_i \circ \pi_i = \pi_i$ and $\Delta_X = \oplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$. The algebraic projectors $\pi_i$ are called the algebraic Künneth projectors.*

**Definition 2.2.** *The motive of $X$ is furthermore said to have a Chow–Künneth decomposition if the algebraic Künneth projectors are orthogonal projectors, i.e., $\pi_i \circ \pi_j = \delta_{i,j}\pi_i$ and $\Delta_X = \oplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$.*

We note that a Chow–Künneth decomposition as above of the diagonal class gives a filtration of the rational Chow groups of $X$, see [Mu2], [Mu3]. Hence it is relevant to look for examples which admit such a decomposition. Our aim in this paper is to investigate this question for rational homogeneous bundles.

## 3. Rational homogeneous bundles over a variety

In this section, we firstly recall the motive of a rational homogeneous variety and later discuss the question of constructing relative Chow–Künneth projectors for a bundle of homogeneous varieties. This is essentially done when the bundle is étale locally trivial over the base. The criterion of [GHMu2] can then be applied to obtain absolute Chow–Künneth projectors on the total space of the bundle.

3.1. **The motive of a rational homogeneous space.** Suppose $F$ is a rational homogeneous space. Then $F = G/P$ is a complete rational variety, for some reductive linear algebraic group $G$ and $P$ is a parabolic subgroup of $G$. Notice that $F$ is a cellular variety, i.e., it has a cellular decomposition

$$\emptyset = F_{-1} \subset F_0 \subset ... \subset F_n = F$$

such that each $F_i \subset F$ is a closed subvariety and $F_i - F_{i-1}$ is an affine space.

Then we have

**Lemma 3.1.** [Ko, Theorem, p.363] *The Chow motive $h(F) = (F, \Delta_F)$ of $F$ decomposes as a direct sum of twisted Tate motives*

$$h(F) = \bigoplus_{\omega} \mathbb{L}^{\otimes dim\ \omega}.$$

*Here $\omega$ runs over the set of cells of $F$.*

In particular, this says that the Chow–Künneth decomposition holds for $F$. Next, we consider bundles of homogeneous spaces $Z \to S$ over a smooth variety $S$. We want to describe the Chow motive of $Z$ in terms of the Chow motive of $S$, upto some Tate twists. This is done under the assumption of étale local triviality which we explain in the next subsection.

3.2. **The étale local triviality of a rational homogeneous bundle.** Consider a rational homogeneous bundle $f : Z \longrightarrow S$, i.e., $\pi$ is a smooth projective morphism and any fibre $\pi^{-1}y$ is a rational homogeneous variety $G/P$. Here $G$ is a reductive linear algebraic group and $P \subset G$ is a parabolic subgroup. Assume that $S$ is a smooth complex projective variety.

By étale local triviality, we mean that there exist étale open covers $p_\alpha : U_\alpha \to S$ such that the pullback bundle

$$Z_{U_\alpha} := Z \times_S U_\alpha \to U_\alpha$$

is a Zariski locally trivial fibration and the images of $p_\alpha$ cover $S$. Here $\alpha$ runs over some indexing set $I$.

In the following discussion, we assume that such a cover exists for $Z \to S$. We want to obtain relative Chow–Künneth projectors for the bundle $Z/S$. For this purpose, we first construct relative projectors over the étale coverings of $Z \to S$ and check the patching conditions. This requires us to use the language of stacks which enables us to descend the projectors down to $Z \to S$. Hence in the following subsection, we recall some facts on regular stacks and the relationship of the rational Chow groups/cohomology of stacks with that of its coarse moduli space. These facts will be essentially applied to the simplest situation—the rational homogeneous bundle $Z \to S$.

3.3. **Chow groups of the regular stack associated to the étale atlas.** Mumford, Gillet ([Mm],[Gi]) have defined Chow groups for Deligne–Mumford stacks and more generally for any algebraic stack $\mathcal{X}$. Furthermore, intersection products are defined whenever $\mathcal{X}$ is a regular stack . Now suppose $\mathcal{X}$ is a regular stack. The coarse moduli space of $\mathcal{X}$ is denoted by $X$ and $p : \mathcal{X} \to X$ be the projection. Then, by [Gi, Theorem 6.8], the pullback $p^*$ and pushforward map $p_*$ establish a ring isomorphism of rational Chow groups

$$CH^*(\mathcal{X})_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}. \tag{1}$$

This can be applied to the product $p \times p : \mathcal{X} \times \mathcal{X} \to X \times X$, to get a ring isomorphism

$$CH^*(\mathcal{X} \times \mathcal{X})_{\mathbb{Q}} \cong CH^*(X \times X)_{\mathbb{Q}}. \tag{2}$$

Assume that $X$ is a smooth projective variety. Then these isomorphisms also hold in the rational singular cohomology of $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ (for example, see [Be]):

$$H^*(\mathcal{X}, \mathbb{Q}) \cong H^*(X, \mathbb{Q}). \tag{3}$$

and

$$H^*(\mathcal{X} \times \mathcal{X}, \mathbb{Q}) \cong H^*(X \times X, \mathbb{Q}). \tag{4}$$

Via these isomorphisms, we can pullback the Künneth decomposition of the diagonal class in $H^{2n}(X \times X, \mathbb{Q})$ to a decomposition of the diagonal class of $\mathcal{X}$ in $H^{2n}(\mathcal{X} \times \mathcal{X}, \mathbb{Q})$, and whose components we refer to as the Künneth components of $\mathcal{X}$.

Given a smooth variety $X$, consider an étale atlas $\sqcup_{\alpha \in I} U_\alpha$ of $X$: here $p_\alpha : U_\alpha \to X$ is a finite étale open cover, for each $\alpha \in I$, and the images of $p_\alpha$ cover $X$. Then one can associate a $Q$-variety [Mm, §2] to this atlas. Furthermore, by [Gi, Proposition 9.2], there is a regular stack $\mathcal{X}$ associated to this data such that $X$ is its coarse moduli space, i.e., there is a projection

$$p : \mathcal{X} \to X.$$

Hence the isomorphisms in (1), (2), (3) and (4) hold for the projection $p$. In the following discussion, we refer to the 'patching conditions over the étale atlas' to be the patching conditions in the associated regular stack. Moreover, we identify the rational Chow groups (respectively cohomology) of an étale atlas $X^{et}$ of a smooth variety $X$ with the rational Chow groups (respectively cohomology) of the associated regular stack $\mathcal{X}$. More precisely, we define

$$CH^*(X^{et})_{\mathbb{Q}} := CH^*(\mathcal{X})_{\mathbb{Q}}$$

and

$$H^*(X^{et}, \mathbb{Q}) := H^*(\mathcal{X}, \mathbb{Q}).$$

3.4. **The motive of a rational homogeneous bundle.** Suppose $Z \to S$ is a rational homogeneous bundle over a smooth projective variety $S$. Let $S^{et} := \sqcup_\alpha U_\alpha$ be an étale atlas of $S$, together with the natural morphism $f : S^{et} \to S$. Here $S$ is considered with the Zariski site. Consider the pullback bundle

$$Z^{et} := Z \times_S S^{et} \to S^{et}$$

over $S^{et}$.

Given any étale open cover $U \to S$, consider the pullback bundle $\pi_U : Z_U \to U$. The rational relative Chow group $CH^*(Z_U/U)_{\mathbb{Q}}$ of $Z_U/U$ is defined as follows:

$$CH^*(Z_U/U)_{\mathbb{Q}} := CH^*(Z_U)/\pi_U^* CH^*(U).$$

Similarly, we define the rational relative Chow groups of $Z^{et} \xrightarrow{\pi} S^{et}$ as

$$CH^*(Z^{et}/S^{et})_{\mathbb{Q}} := CH^*(Z^{et})_{\mathbb{Q}}/\pi^* CH^*(S^{et})_{\mathbb{Q}}.$$

Notice that these relative groups can also be defined for other cohomology theories. In particular, for the singular cohomology theory and we denote the relative singular cohomology groups by $H^*(Z/S, \mathbb{Q})$ and $H^*(Z^{et}/S^{et}, \mathbb{Q})$ of $Z/S$ and $Z^{et}/S^{et}$ respectively.

Since we are dealing with a rational homogeneous bundle, we can describe these groups explicitly as follows; by assumption, the pullback bundles $Z_{U_\alpha} \to U_\alpha$, for $\alpha \in I$, are Zariski locally trivial. Hence $Z_{U_\alpha} \to U_\alpha$ is a relative cellular variety, for each $\alpha \in I$.

The description of the rational Chow groups of a relative cellular space $f : X \to T$ is given by B. Koeck [Ko] (see also [Ne-Za, Theorem 5.9]), which is stated for the higher Chow groups:

Suppose $X \to T$ is a relative cellular space.

Then there is a sequence of closed embeddings

$$\emptyset = Z_{-1} \subset Z_0 \subset ... \subset Z_n = X \tag{5}$$

such that $\pi_k : Z_k \longrightarrow T$ is a flat projective $T$-scheme. Furthermore, for any $k = 0, 1, ..., n$, the open complement $Z_k - Z_{k-1}$ is $T$-isomorphic to an affine space $\mathbb{A}_T^{m_k}$ of relative dimension $m_k$. Denote $i_k : Z_k \hookrightarrow X$.

**Theorem 3.2.** *For any $a, b \in \mathbb{Z}$, the map*

$$\begin{array}{ccc} \bigoplus_{k=0}^{n} H_{a-2m_k}(T, b - m_k) & \longrightarrow & H_a(X, b) \\ (\alpha_0, ..., \alpha_n) & \mapsto & \sum_{k=0}^{n} (i_k)_* \pi_k^* \alpha_k \end{array}$$

*is an isomorphism. Here $H_a(T, b) = CH_b(T, a - 2b)$ are the higher Chow groups of $T$.*

*Proof.* See [Ko, Theorem, p.371]. □

The above theorem can equivalently be restated to express the rational Chow groups of $X$ as

$$CH^r(X)_{\mathbb{Q}} = \bigoplus_{k=0}^{r} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).f^*CH^k(T)_{\mathbb{Q}}. \tag{6}$$

Here $\omega_k^\gamma$ are the $r-k$ codimensional relative cells and $\gamma$ runs over the indexing set of $r-k$ codimensional relative cells in the $T$-scheme $X$.

We now apply this theorem to our situation: we have a homogeneous bundle $Z \to S$ and an étale atlas $S^{et} := \sqcup_\alpha U_\alpha \to S$, such that $Z_{U_\alpha} \to U_\alpha$ is locally trivial.

**Lemma 3.3.** *Given a Zariski locally trivial homogeneous bundle $p_\alpha : Z_{U_\alpha} \to U_\alpha$, the relative rational Chow groups are described as follows:*

$$CH^r(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} = \bigoplus_{k=0}^{r-1} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).p_\alpha^* CH^k(U_\alpha)_{\mathbb{Q}}.$$

*Proof.* Since the homogeneous bundle $p_\alpha : Z_{U_\alpha} \to U_\alpha$ is a Zariski locally trivial bundle, it is a relative cellular variety. Hence the above Theorem 3.2 can be applied and it gives a splitting of the quotient map

$$CH^*(Z_{U_\alpha})_{\mathbb{Q}} \to CH^*(Z_{U_\alpha})_{\mathbb{Q}}/p_\alpha^* CH^*(U_\alpha)_{\mathbb{Q}} =: CH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}.$$

This gives a natural isomorphism

$$CH^r(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} = \bigoplus_{k=0}^{r-1} (\oplus_\gamma \mathbb{Q}[\omega_k^\gamma]).p_\alpha^* CH^k(U_\alpha)_{\mathbb{Q}}.$$

□

For our applications, it suffices to consider the piece $k = 0$, which consists of only the relative algebraic cells of codimension $r$, namely,

$$RCH^r(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma].$$

Similarly, the result of [Ko], [Ne-Za] holds in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$. So we can also define the piece

$$RH^{2r}(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$ and the piece

$$RH^{2r}(Z/S)_{\mathbb{Q}} := \oplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

as a subspace of the rational Betti cohomology $H^{2r}(Z, \mathbb{Q})$, generated by the relative analytic cells $\omega_0^\gamma$. Here, we use the fact that $Z \to S$ is locally trivial in the analytic topology and there is a analytic cellular decomposition similar to (5).

**Lemma 3.4.** *The cycles $\omega_0^\gamma$ in $RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}$ patch together for the étale atlas to determine a subspace $RCH^*(Z^{et}/S^{et})_{\mathbb{Q}}$ of $CH^*(Z^{et})_{\mathbb{Q}}$, generated by the patched cycles. This subspace maps isomorphically onto the subspace $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z,\mathbb{Q})$, under the cycle class map*

$$CH^*(Z^{et})_{\mathbb{Q}} \to H^{2*}(Z^{et},\mathbb{Q}) \simeq H^{2*}(Z,\mathbb{Q}).$$

*Proof.* Using the isomorphism in (3), there is an isomorphism

$$H^{2*}(Z^{et},\mathbb{Q}) \simeq H^{2*}(Z,\mathbb{Q}).$$

Hence the cycles $\omega_0^\gamma \in RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}}$ patch together as analytic cycles for the étale atlas and determine a subspace $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}} \subset H^{2r}(Z^{et},\mathbb{Q})$, mapping isomorphically onto $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z,\mathbb{Q})$.

Now, by definition, there is a natural isomorphism

$$RCH^*(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} \xrightarrow{\simeq} RH^{2*}(Z_{U_\alpha}/U_\alpha)_{\mathbb{Q}} \tag{7}$$

between the 0-th piece of the rational relative Chow groups and the relative Betti cohomology, for each $\alpha$.

Via the isomorphism in (7), the patching conditions required for the étale atlas, to define the piece $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$ are the same as those for $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$. More precisely, the patching conditions are given in [Gi, §4]. The identification in (7) together with the fact that the patching conditions are fulfilled for the singular cohomology for the étale atlas, says that the cycles $\omega_0^\gamma$ patch together to give a class in $RH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$. Hence they also patch together to give a class in $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}}$. These patched classes generate the $\mathbb{Q}$-subspace $RCH^{2r}(Z^{et}/S^{et})_{\mathbb{Q}} \subset CH^*(Z^{et})_{\mathbb{Q}}$ and which maps isomorphically onto the subspace $RH^{2r}(Z/S)_{\mathbb{Q}} \subset H^{2r}(Z,\mathbb{Q})$ under the cycle class map. □

**Corollary 3.5.** *There is a canonical isomorphism*

$$RCH^r(Z^{et}/S^{et})_{\mathbb{Q}} \simeq RH^{2r}(Z/S)_{\mathbb{Q}}.$$

*between the relative Chow groups and the relative rational cohomology generated by the relative cells.*

□

Let $n := \dim(Z/S)$.

**Corollary 3.6.** *The bundle $Z \to S$ has a relative Chow–Künneth decomposition, in the sense of* [GHMu].

*Proof.* This is an application of Lemma 3.4. We apply the ring isomorphisms of (3), (4) to the relative groups of the product spaces $(Z^{et} \times_{S^{et}} Z^{et}) \to S^{et}$, $(Z \times_S Z) \to S$ and notice that the relative orthogonal Künneth projectors in $H^{2n}(Z \times_S Z/S,\mathbb{Q})$ lift to orthogonal projectors in $H^{2n}(Z^{et} \times_{S^{et}} Z^{et}/S^{et},\mathbb{Q})$ and which add to the relative diagonal cycle. Now we note that the relative diagonal $\Delta_{Z/S}$ and its orthogonal Künneth components actually lie in the piece $RH^{2n}(Z \times_S Z/S)_{\mathbb{Q}}$ (generated by the relative algebraic cells) and under the isomorphisms in (3), (4), lift to an orthogonal decomposition

$$\Delta_{Z^{et}/S^{et}} = \sum_{i=0}^{2n} \Pi_i \in RH^{2n}(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$$

over the étale atlas. Now apply Corollary 3.5 to the product space $Z^{et} \times_{S^{et}} Z^{et} \to S^{et}$, to lift the above orthogonal projectors to orthogonal algebraic projectors in $RCH^n(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$, and which add to the relative diagonal cycle $\Delta_{Z^{et}/S^{et}}$ in $CH^n(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$. Now, the ring isomorphism in (2) says that the rational Chow groups of the étale atlas and the rational Chow groups of the Zariski site are isomorphic and the relative orthogonal algebraic projectors $\Pi_i \in RCH^n(Z^{et} \times_{S^{et}} Z^{et}/S^{et})_{\mathbb{Q}}$ descend to the rational Chow groups of $(Z \times_S Z)/S$. This gives relative Chow–Künneth projectors and a relative Chow–Künneth decomposition

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \in CH^n(Z \times_S Z)_{\mathbb{Q}}.$$

□

**Proposition 3.7.** *Suppose $Z \to S$ is a rational homogeneous bundle over a smooth variety $S$, which is locally trivial over the étale atlas of $S$. Then the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of $S$ with the twisted Tate motive. More precisely, the motive of $Z$ can be written as*

$$h(Z) = \bigoplus_i h^i(Z)$$

*where $h^i(Z) = \bigoplus_{j+k} r_{\omega_\alpha}.\mathbb{L}^j \otimes h^k(S)$. Here $r_{\omega_\alpha}$ is the number of $j$-codimensional cells on a fiber $\mathbb{F}$*

*In particular, if $S$ has a Chow–Künneth decomposition then $Z$ also admits an absolute Chow–Künneth decomposition.*

*Proof.* By Corollary 3.6, we know that the bundle $Z/S$ has a relative Chow–Künneth decomposition. Since the map $Z \to S$ is a smooth morphism and the fibers of $Z \to S$ have only algebraic cohomology, we can directly apply the criterion in [GHMu2, Main theorem 1.3], to get absolute Chow–Künneth projectors for $Z$ and the decomposition stated above (for example, see [Iy, Lemma 3.2, Corollary 3.3]). □

**Remark 3.8.** *Suppose $X$ is a smooth projective variety with a nef tangent bundle. Then by* [Ca-Pe],[DPS]*, we know that there is an étale cover $X' \to X$ of $X$ such that $X' \to A$ is a smooth morphism over an abelian variety $A$, whose fibers are smooth Fano varieties with a nef tangent bundle. It is an open question* [Ca-Pe, p.170]*, whether such a Fano variety is a rational homogeneous variety. A positive answer to this question, together with Proposition 3.7, will give absolute Chow–Künneth projectors for a class of varieties with a nef tangent bundle.*

## References


[Ak-Jo] Akhtar, R., Joshua, R. *Künneth decomposition for quotient varieties*, Künneth decompositions for quotient varieties. Indag. Math. (N.S.) 17 (2006), no. **3**, 319–344.

[Ar-Dl] Arapura, D., Dhillon, A. *The motive of the moduli stack of G-bundles over the universal curve*, Proc. Indian Acad. Sci. Math. Sci. 118 (2008), no. **3**, 389–411.

[Be] Behrend, K. *On the de Rham cohomology of differential and algebraic stacks.* Adv. Math. 198 (2005), no. **2**, 583–622.

[Ca-Pe] Campana, F. and Peternell, T. *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann. **289** (1991), 169-187.

[dA-Ml] del Angel, P., Müller–Stach, S. *Motives of uniruled* 3*-folds*, Compositio Math. 112 (1998), no. **1**, 1–16.

[dA-Ml2] del Angel, P., Müller-Stach, S. *On Chow motives of 3-folds*, Trans. Amer. Math. Soc. 352 (2000), no. **4**, 1623–1633.

[DPS] Demailly, J.P, Peternell, T., Schneider, M. *Compact complex manifolds with numerically effective tangent bundles*, Journal of Algebraic Geometry **3** (1994), 295-345.

[De-Mu] Deninger, Ch., Murre, J. *Motivic decomposition of abelian schemes and the Fourier transform*, J. Reine Angew. Math. **422** (1991), 201–219.

[Gi] Gillet, H. *Intersection theory on algebraic stacks and $Q$-varieties*, Proceedings of the Luminy conference on algebraic $K$-theory (Luminy, 1983). J. Pure Appl. Algebra **34** (1984), 193–240.

[Go-Mu] Gordon, B., Murre, J. *Chow motives of elliptic modular threefolds*, J. Reine Angew. Math. **514** (1999), 145–164.

[GHMu] Gordon, B. B., Hanamura, M., Murre, J.P. *Relative Chow-Künneth projectors for modular varieties* J. Reine Angew. Math. **558** (2003), 1–14.

[GHMu2] Gordon, B. B., Hanamura, M., Murre, J. P. *Absolute Chow-Künneth projectors for modular varieties*, J. Reine Angew. Math. **580** (2005), 139–155.

[Gu-Pe] Guletskiĭ, V., Pedrini, C. *Finite-dimensional motives and the conjectures of Beilinson and Murre* Special issue in honor of Hyman Bass on his seventieth birthday. Part III. $K$-Theory 30 (2003), no. **3**, 243–263.

[Iy] Iyer, J.N., *Murre's conjectures and explicit Chow Künneth projectors for varieties with a nef tangent bundle*, Trans. Amer. Math. Soc. 361 (2009), no. **3**, 1667–1681.

[Iy-Ml] Iyer, J.N., Müller-Stach, S. *Chow–Künneth decomposition for some moduli spaces*, arXiv math.AG/07104002, to appear in Documenta Mathematica.

[Ja] Jannsen, U. *Motivic sheaves and filtrations on Chow groups*, Motives (Seattle, WA, 1991), 245–302, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Ko] Köck, B. *Chow motif and higher Chow theory of $G/P$*, Manuscripta Math. **70** (1991), 363–372.

[Mn] Manin, Yu. *Correspondences, motifs and monoidal transformations* (in Russian), Mat. Sb. (N.S.) **77** (119) (1968), 475–507.

[MWYK] Miller, A., Müller-Stach, S., Wortmann, S., Yang, Y.H., Kang Z. *Chow-Künneth decomposition for universal families over Picard modular surfaces*, Algebraic cycles and motives. Vol. **2**, 241–276, London Math. Soc. Lecture Note Ser., 344, Cambridge Univ. Press, Cambridge, 2007.

[Mi] Miller, A. *Chow motives of universal families over some Shimura surfaces*, arXiv math.AG/0710.4209.

[Mm] Mumford, D. *Towards an Enumerative Geometry of the Moduli Space of Curves*, Arithmetic and geometry, Vol. II, 271–328, Progr. Math., **36**, Birkhäuser Boston, Boston, MA, 1983.

[Mu] Murre, J. P. *On the motive of an algebraic surface*, J. Reine Angew. Math. **409** (1990), 190–204.

[Mu2] Murre, J. P. *On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples*, Indag. Math. (N.S.) 4 (1993), no. **2**, 177–188.

[Mu3] Murre, J. P. *On a conjectural filtration on the Chow groups of an algebraic variety. II. Verification of the conjectures for threefolds which are the product on a surface and a curve*, Indag. Math. (N.S.) 4 (1993), no. **2**, 189–201.

[Ne-Za] Nenashev, A., Zainoulline, K. *Oriented cohomology and motivic decompositions of relative cellular spaces*, J. Pure Appl. Algebra 205 (2006), no. **2**, 323–340.

[Sa] Saito, M. *Chow-Künneth decomposition for varieties with low cohomological level*, arXiv math.AG/0604254.

[Sc] Scholl, A. J. *Classical motives*, Motives (Seattle, WA, 1991), 163–187, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Sh] Shermenev, A.M. *The motive of an abelian variety*, Funct. Analysis, **8** (1974), 55–61.

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