# On Non-Relativistic Limits

# Of the AAdS-CFT Conjecture


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# Based on

- Arjun Bagchi and Rajesh Gopakumar, (to appear )
- Arjun Bagchi, Turbasu Biswas and R. Gopakumar (work in progress).

# Outline of the talk

- Motivation and General Remarks
  - Non-Relativistic Conformal Symmetry
- The Non-Relativistic Contraction
  - And a natural *infinite* dimensional extension
- Bulk Realisation of the Contraction
  - Galilean limits of Einstein's Equations
- "To Do" List

# 1 Motivation and General Remarks

# Motivation

- Recent thrust towards understanding non-relativistic systems (e.g. in CMT) in the same spirit as the (relativistic) AdS/CFT correspondence.
- However, mostly in the context of the non-relativistic Schrodinger Symmetry (cf. talk by Kaushik).
- Are there other non-relativistic symmetries? Yes.
- One way is to take a direct non-relativistic limit of the relativistic theory.
- Also opens the possibility of studying a BMN like limit of the parent theory (e.g. $\mathcal{N} = 4$ Yang-Mills theory).
- Thus focus on a potentially simpler subsector of the full hilbert space.
- Seems to be much more constrained by symmetry (Kac-Moody Algebra).
- Report some preliminary progress along these lines.

# General Remarks

- Schrodinger Symmetry $Sch(d,1)$ is a non-relativistic analogue of conformal symmetry in $(d+1)$ dimensions.

- It contains the usual Galilean group $G(d,1)$ (with a central extension):

$$\begin{aligned} [J_{ij}, J_{rs}] &= so(d) \\ [J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\ [J_{ij},\, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij},\, H] = 0 \\ [B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = m\delta_{ij} \\ [H, P_i] &= 0, \quad [H, B_i] = -P_i. \end{aligned} \tag{1}$$

- But also has *two* more generators $\tilde{K}, \tilde{D}$ with some of its commutators given by

$$\begin{aligned} [\tilde{K}, P_i] &= B_i, \quad [\tilde{K}, B_i] = 0, \quad [\tilde{D}, B_i] = B_i \\ [\tilde{D}, \tilde{K}] &= 2\tilde{K}, \quad [\tilde{K}, H] = -\tilde{D}, \quad [\tilde{D}, H] = -2H. \end{aligned} \tag{2}$$

- The last line is an $SL(2,R)$ algebra.

# General Remarks *continued*

- $\tilde{K}$ is like a time component of special conformal transformations.
- No analogue of the spatial components $K_i$ of special conformal transformations.
- Thus 12 parameter group (+ central term) in $d = 3$ as opposed to 15 parameters for $SO(4, 2)$.
- $\tilde{D}$ is a dilatation operator which scales time and space *differently*

$$x_i \to \lambda x_i, t \to \lambda^2 t. \tag{3}$$

- $Sch(d, 1)$ is the symmetry of the free particle Schrodinger equation, i.e. generators that commute with the Schrodinger operator $S = i\partial_t + \frac{1}{2m}\partial_i^2$.
- This symmetry also believed to arise in interacting systems like fermions at unitarity.

# 2 The Non-Relativistic Contraction

# Galilean Contraction

- A direct way to recover the Galilean group $G(d,1)$ is to perform a contraction on the Poincare group $ISO(d,1)$.
- Take $t \to \epsilon^r t$ and $x_i \to \epsilon^{r+1} x_i$ and scale $\epsilon \to 0$. (Thus $v_i \sim \epsilon$).
- The vector fields generating the Poincare algebra reduce to the Galilean vector fields (after appropriate rescaling)

$$\begin{aligned} H &= -\partial_t, \quad P_i = \partial_i \\ B_i &= t\partial_i, \quad M_{ij} = -(x_i\partial_j - x_j\partial_i). \end{aligned} \tag{4}$$

- One might therefore try and do a similar contraction on the relativistic conformal group $SO(d,2)$ to obtain a non-relativistic conformal group.
- Since this is obtained as a parametric scaling limit, it is directly embedded within the relativistic theory.

- Carry out this scaling on the additional generators $(D, K_0, K_i)$ present in the conformal group $SO(d, 2)$.

- Gives the contracted vector fields

$$\begin{aligned} D &= -(x_i\partial_i + t\partial_t) \\ K &= -(2tx_i\partial_i + t^2\partial_t) \\ K_i &= t^2\partial_i. \end{aligned} \tag{5}$$

- Note the dilatation generator $D$ is the *same* as in the relativistic theory. It scales space and time in the same way.

- Therefore different from $\tilde{D} = -(2t\partial_t + x_i\partial_i)$.

- Spatial conformal transformation generators $K_i$ *are present* and generate constant acceleration transformations.

- The temporal special conformal generator different from $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$

# Galilean Contraction *Conformal Generators*

- The algebra of these generators (together with that of the Galilean Algebra) is quite different from the Schrodinger group.

- However, a subset of these generators appear as symmetries of Navier-Stokes equations (cf. talk by Spenta).

- Galilean central extension in $[B_i, P_j]$ not admissible in this algebra – "massless non-relativistic system".

- However, one can have a different central extension of the form $[K_i, P_j] = N\delta_{ij}$. (Interpretation?)

- Algebra of the contracted conformal group: Define

$$
\begin{aligned}
L^{(-1)} &= H, \qquad L^{(0)} = D, \qquad L^{(+1)} = K, \\
M_i^{(-1)} &= P_i, \qquad M_i^{(0)} = B_i, \qquad M_i^{(+1)} = K_i.
\end{aligned}
\tag{6}
$$

- Then

$$
\begin{aligned}
[J_{ij}, L^{(n)}] &= 0, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\
[J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), \qquad [M_i^{(m)}, M_j^{(n)}] = 0, \\
[L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}.
\end{aligned}
\tag{7}
$$

- Note the $SL(2,R)$ algebra in the last line. *Different* from that in the Schrodinger group.

# Galilean Contraction *Infinite dimensional extension*

- The most remarkable feature of this algebra is that it admits a very natural extension to an infinite dimensional $SO(d)$ Current Algebra.

- Define the vector fields for arbitrary integer $n$

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \\ M_i^{(n)} &= t^{n+1}\partial_i \\ J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i) \end{aligned} \tag{8}$$

- They obey exactly the same commutation relations as the ones for $m, n = 0, \pm 1$.

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \end{aligned} \tag{9}$$

- The Virasoro and Kac-Moody algebra of the vector fields is, of course, without the central extension.

# The Galilean Contraction *Interpretation*

- The $M_i^{(n)}$ act as generators of generalised time dependent but spatially homogeneous accelerations

$$x_i \to x_i + b_i(t). \tag{10}$$

- Similarly, the $J_{ij}^{(n)} \equiv J_a^{(n)}$ are generators of arbitrary time dependent rotations

$$x_i \to R_{ij}(t)x_j \tag{11}$$

- These two together generate what is sometimes called the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime.
- $L^{(n)}$ seem to be generators of a conformal "isometry" of Galilean spacetime.

$$t \to f(t), \qquad x_i \to \frac{df}{dt}x_i \tag{12}$$

- Galilean spacetime (in the absence of gravitation) is characterised by a degenerate metric: an absolute time $(t)$ and a Euclidean spatial metric.

# The Galilean Contraction *Realisations*

- How are these Galilean limits realised in gauge theories?
- One would expect an $SL(2, R)$ sector of the relativistic gauge theory in which the spatial rotation group behaves like an internal symmetry.
- Bulk dual provides hints as to how to look for this (see later).
- Another possibility is a partial realisation of a subset of these symmetries.
- For gauge theories at finite temperature in the hydrodynamic limit, one can study the non-relativistic limit (Fouxon and Oz, Bhattacharya, Minwalla and Wadia).
- Recover the Navier-Stokes equations. Has all the symmetries of the *finite* contracted algebra (except $D$ which is broken by choice of temperature/scale $T$).
- Actually also has the whole of the boosts $M_i^{(n)}$ (Milne algebra).

# 3 Bulk Dual of the Galilean Conformal Algebra

# The Bulk Dual

- What can we say about the bulk dual to a system with this Galilean Conformal Algebra?

- The $SL(2,R)$ suggests an $AdS_2$ part: in the $t$ and $z$ (radial directions).

- Boundary metric degenerating in the non-relativistic limit $\Rightarrow$ some kind of Galilean limit in the $(d+2)$ dimensional bulk.

- (Alternatively, realise the non-relativistic symmetry in a one higher dimensional bulk?)

- So expect the dual spacetime to have the structure $AdS_2 \times R^3$ with degenerate metric on the $R^3$.

- Will first check how the (extended) Galilean Conformal Algebra is realised in this kind of bulk. And outline the geometric structure underlying it (analogous to Newtonian limit of Einstein's equations).

# The Bulk Dual *Symmetries*

- $AdS_{d+2}$ in Poincare coordinates:

$$ds^2 = R^2 \frac{dt^2 - dz^2 - dx_i^2}{z^2} \tag{13}$$

- In radially infalling coordinates for null geodesics $(t' = t + z, z' = z)$

$$ds^2 = \frac{R^2}{z'^2}(-2dt'dz' + dt'^2 - dx_i^2) = \frac{R^2}{z'^2}(-dt'(2dz' - dt') - dx_i^2). \tag{14}$$

- Take the generators of the $AdS_{d+2}$ isometries and perform the contraction by taking $t', z^{'} \to \epsilon^r, x_i \to \epsilon^{r+1} x_i$. Metric degenerates as expected.

- Contracted Killing vectors given by

$$\begin{aligned} P_i &= -\partial_i, \quad B_i = -(t' - z')\partial_i, \quad K_i = -(t'^2 - 2t'z')\partial_i \\ H &= \partial_{t'}, \quad D = t'\partial_{t'} + z'\partial_{z'} + x_i\partial_i, \quad K = t'^2\partial_{t'} + 2(t' - z')(z'\partial_{z'} + x_i\partial_i). \end{aligned} \tag{15}$$

# The Bulk Dual *Symmetries*

★ More compactly (for $m, n = 0, \pm 1, l = 0$).

$$
\begin{aligned}
L^{(n)} &= t'^{n+1}\partial_{t'} + (n+1)(t'^{n} - nzt'^{n-1})(x_i\partial_i + z'\partial_{z'}) \\
M_i^{(m)} &= -(t'^{m+1} - (m+1)zt'^{m})\partial_i \\
J_{ij}^{(l)} &= -t'^{n}(x_i\partial_j - x_j p_i)
\end{aligned} \tag{16}
$$

★ Reduces at the boundary ($z = 0$) to the generators of the contracted conformal algebra. And satisfies the *same* algebra.

★ In fact, these bulk vector fields (for arbitrary $m, n, l$) reduce to that of the extended Kac-Moody algebra at the boundary.

★ What is the role of these vector fields in the bulk?

★ The Virasoro generators act as the generators of asymptotic symmetries of the $AdS_2$.

★ The others act only on the $R^3$ (like Galilean "isometries" on the boundary).

# The Bulk Dual *Geometric Structure*

- ★ Degenerate nature of the metric might seem to imply that the gravitational dynamics is singular.
- ★ However, similar situation in asymptotically flat space in recovering Newtonian gravity from Einstein gravity in the non-relativistic limit.
- ★ The answer: there is a well-defined geometric theory of newtonian gravitation - Newton-Cartan theory (see for e.g. MTW).
- ★ The ingredients: A space time endowed with absolute time function $t$, a non-dynamical *spatial* (Euclidean) metric, a dynamical *non-metric* connection $\Gamma^i_{00} \propto \partial_i \Phi$.
- ★ Einstein's equations reduce to $R_{00} \propto \rho$ which is Poisson's equation for $\Phi$.
- ★ One can write all this in a more covariant form: A three dimensional fibre $R^3$ over a base $R$ which is parametrised by the time $t$.
- ★ Nothing singular about this classical geometric description, only unusual.

# The Bulk Dual *Geometric Structure*

★ Proposal: Proceed here in a similar manner.

★ Except that instead of base $R$ we have $AdS_2$ and fibres are still Euclidean $R^3$.

★ Separate metrics $g_{\alpha\beta}$ on $AdS_2$ and $\delta_{ij}$ on the spatial $R^3$.

★ Dynamical affine connections $\Gamma^i_{\alpha\beta}$.

★ The non-relativistic scaling limit of Einstein's equations leave one with

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{17}$$

★ Thus the bulk boundary relation is some kind of an $AdS_2/CFT_1$ duality.

★ Presumably existing inside every relativistic conformal field theory in higher dimensions.

# 4 "To Do" List

# "To Do" List

- ☆ Full realisation of galilean limit within relativistic gauge theories, (*a la* BMN) e.g. in $\mathcal{N} = 4$ SYM.
- ☆ Realisation/calculation of central charges in individual systems (*a la* Brown-Henneaux
- ☆ How far can one exploit the presence of this Kac-Moody algebra sitting in higher dimensional gauge theories?
- ☆ Supersymmetric generalisation.
- ☆ Understand bulk geometric description better. Closed string description of the geometry?
- ☆ Develop the bulk-boundary dictionary.
- ☆ Is there a more standard geometric description in, say, $(d+3)$ dimensions as in the bulk duals for Schrodinger symmetric theories?
- ☆ Are there intrinsically non-relativistic realisations of Galilean conformal algebra?

# The end