Monodromy and Arithmetic Groups

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It is easy to show that there are many thin groups but hard to tell if a *given* group is one. In general it is impossible to decide if a group Γ is thin or arithmetic.
Examples of arithmetic groups: $SL_n(\mathbb{Z})$, $Sp_{2g}(\mathbb{Z})$ or subgroups of finite index in them.

Restricton of Scalars

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Similarly, if $\mathcal{G}$ is a linear algebraic group defined over a number field, there is a linear algebraic group G defined over $\mathbb{Q}$ such that $G(\mathbb{R}) = \mathcal{G}(K \otimes_{\mathbb{Q}} \mathbb{R})$ and $G(\mathbb{Z}) \simeq \mathcal{G}(O_K)$, where O_K is the ring of integers in the number field K. G is group obtained from $\mathcal{G}$ by the "Weil restriction of scalars" from K to $\mathbb{Q}$. The sign $\simeq$ means that the groups are *commensurable*.

Monodromy Groups

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The image of this representation is called the *monodromy* of the fibration.

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However, this conjecture is false in general and it is not clear what are the (geometric) hypotheses to be made on the fibration , which ensure that the monodromy is arithmetic. We first look at cases where the monodromy is arithmetic.

An Example

Take $S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. For X, we take the Legendre family of of elliptic curves $X = \{(x, y, \lambda) : y^2 = x(x-1)(x-\lambda), \quad \lambda \in S$ with the fibration being the projection to the λ part .

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The fundamental group of S is the free group on two generators, h_0, h_∞ given by small loops going around 0 and ∞.

The monodromy in this case is the representation $\pi_1(S) \to GL(H^1(F, \mathbb{Z})) = GL_2(\mathbb{Z})$ given by $h_0 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $h_\infty \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. These two matrices generate the principal congruence subgroup of level 2 in $SL_2(\mathbb{Z})$. In particular, the monodromy is arithmetic.

A'Campo's Theorem

Take for S the space $f = (x - a_1) \cdots (x - a_n)$ of monic polynomials of degree n with distinct roots. Consider the family $\{(y, x, f) : y^2 = f(x)\}$ of hyperelliptic curves. All these curves have the same genus g and the monodromy group preserves a symplectic form (the intersection form) on H^1 of these curves.

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The fibres of the family $X \to S$ are double covers of the projective line which ramify at n distinct points and possibly at infinity.

The Main Result

We keep the same base S of monic polynomials f of degree n with distinct roots, but consider the family $X = \{(x, y, f) : y^d = f(x)\}$ for a fixed integer $d \geq 3$. The fibres are cyclic covers of degree d of the projective line ramified at n distinct points and maybe at infinity.

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The generator T of the group $\mathbb{Z}/d\mathbb{Z}$ acts on the cyclic cover $y^d = f(x)$ by sending y to $y\omega$ for a primitive d-th root of unity ω. It is easy to see that the monodromy action commutes with the action of T, and hence $\Gamma \subset Sp_{2g}(\mathbb{Z})^T$, the subgroup of the symplectic group which commutes with T.

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Theorem

(T.N.Venkataramana, Annals of Math 179 (2014) The monodromy of this family is arithmetic, provided $n \geq d + 1$. More precisely, Γ has finite index in $Sp_{2g}(\mathbb{Z})^T$.

If we take $d = 18$ and $n = 4$, then the monodromy is thin (i.e. not arithmetic) by the theory of Deligne and Mostow. Therefore, some condition on n, d is certainly needed for monodromy to be arithmetic.

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Corollary

If $d = 3, 4, 6$ then the monodromy is arithmetic for all n.

General Cyclic covers

Let $d \geq 2$ and $1 \leq k_1, \cdots, k_n \leq d-1$ and be integers and suppose $d, k_1, \cdots, k_n$ are coprime. Consider the open curve $C = C_a : y^d = (x-a_1)^{k_1} \cdots (x-a_n)^{k_n}$ where a_i are distinct complex numbers. The curve C_a is a compact Riemann surface F_a with finitely many punctures. The genus g of F_a is fixed independent of a.

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Take for S the open subset of $\mathbb{C}^n$ whose entries are all distinct. Then the monodromy of the family X of the compact Riemann surfaces F_a lies in $Sp_{2g}(\mathbb{Z})$.

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Theorem

(T.N.Venkataramana, Invent Math, 197 (2014)) If all the k_i are co-prime to d, and $n \geq 2d+1$ then the foregoing monodromy is arithmetic.

As before, it is not true in general that the monodromy is arithmetic (in the range $n \leq d$). There is a general criterion by Deligne-Mostow which tells us when the monodromy is not arithmetic (i.e. is thin).

The fundamental group of the base

The monodromy in our situation acts on $H^1_{\mathbb{C}} = H^1$ with $\mathbb{C}$-coefficients, of the curve (the fibre) and we need to understand the fundamental group of the base as well as the irreducible components of the representation. The base S is the space of monic polynomials of degree n with distinct roots.

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One can show that $\pi_1(S)$ is the Artin braid group B_n on n-strands. It has generators $s_i(1 \leq i \leq n-1$ with relations $s_i s_j = s_j s_i \quad | i-j | \geq 2)$ and the braiding relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n-2$.

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The braid group B_n acts on $H^1_{\mathbb{C}}$. The latter has an action by $T \in \mathbb{Z}/d\mathbb{Z}$, and splits into ($B_n$-stable) eigenspaces V_f with eigenvalue ω^f, for each $f \in \mathbb{Z}/d\mathbb{Z}$. The intersection form on H^1 extends to a Hermitian form h on $H^1_{\mathbb{C}}$. Suppose the signature of the restriction of h to V_f is (p_f, q_f). Thus $Sp^T_{2g}(\mathbb{R}) = \prod_f U(p_f, q_f)$.

The Burau Representation

We define the Burau representation $\rho : B_n \to GL_{n-1}(\mathbb{Z}[q, q^{-1}])$. Let $R = \mathbb{Z}[q, q^{-1}]$ and R^{n-1} the free R module of rank $n-1$ with standard basis $e_1, \cdots, e_{n-1}$. B_n has the standard generators $s_1, \cdots, s_{n-1}$. Each s_i acts on e_j as follows. $s_i(e_j) = e_j$ if $\mid j - i \mid \geq 2$. $s_i(e_i) = -qe_i$, $s_i(e_{i-1}) = e_{i-1} + qe_i$, $s_i(e_{i+1}) = e_i + e_{i+1}$. Restricted to the submodule generated by e_{i-1}, e_i, e_{i+1}, the matrix of s_i has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ q & -q & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Suppose q is specialised to the d-th root of unity ω^f. We then get the Burau representation evaluated at ω^f, as the specialisation $\rho_f : B_n \to GL_{n-1}(R) \to GL_{n-1}(\mathbb{Z}[\omega^f])$.

Monodromy and Burau representation

The Burau representation ρ is absolutely irreducible. When specialised to a primitive d-th root of unity, it (i.e. ρ_d) continues to be irreducible, except when d divides n, in which case there is a one dimnesional space $\mathbb{C}v$ generated by an invariant vector and the quotient $\overline{\rho}_d$ by this line is irreducible. We write $\overline{\rho}_d = \rho_d$ even when ρ_d is irreducible.

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The monodromy representation is the direct sum $\oplus\overline{\rho}_e$ where the sum runs through all the divisors $e \neq 1$ of d.

A Hermitian form on the Burau representation

The ring $R = \mathbb{Z}[q, q^{-1}]$ has an involution defined by sending q to its inverse. On the free R module R^{n-1} (with respect to the standard basis e_i), we define a Hermitian form h by setting $h(e_i, e_j) = 0$ if $\mid i - j \mid \geq 2$, $h(e_i, e_i) = \frac{(q+1)^2}{q}$ and $h(e_i, e_{i+1}) = \frac{q+1}{q}$.

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The braid group B_n preserves this Hermitian form under the Burau action, and hence $\rho : B_n \to U(h)(R)$, the unitary group of the form h. We also get, by composition, the representation $\rho_d : B_n \to U(h)(\mathbb{Z}[\omega])$ where ω is a primitive d-th root of unity. The latter ring $O_d = \mathbb{Z}[\omega]$ is the ring of integers in the d-th cyclotomic extension $\mathbb{Q}(e^{2\pi i/d})$ of $\mathbb{Q}$.

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The arithmeticity of the monodromy follows from the

Theorem

If $n \geq 2d + 1$, then the image of ρ_d has finite index in the arithmetic group $U(h)(O_d)$.

Criterion for thin-ness

Before describing the proof, we describe why the condition $n \geq 2d + 1$ appears. Return to the definition of a thin group. Under certain conditions, the monodromy group $\Gamma \subset GL_N(\mathbb{Z})$ is thin.

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If we can show that one of the projections of Γ has discrete image, then it follows that Γ cannot have finite index, i.e. Γ is thin. This is the strategy of Deligne-Mostow.

Deligne-Mostow

Consider the family (as the a_i vary while remaining distinct from each other, of curves $y^d = (x-a_1)^{k_1} \cdots (x-a_n)^{k_n}$, with $1 \leq k_i \leq d-1$. Since $\mathbb{Z}/d\mathbb{Z}$ acts on H^1 of this curve, we may consider the "$\mathbb{Z}/d\mathbb{Z}$-primitive part of H^1 which, by definition, is a sum of eigenspaces M_f for the generator T of the cyclic group, with eigenvalues *primitive* d-th roots of unity of the form ω^f .

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The monodromy Γ lies in $U(h)(O_d)$ where h is the intersection form, and O_d is the ring of integers in the d-th cyclotomic extension of $\mathbb{Q}$. Hence $U(h)(K \otimes \mathbb{R}) = \prod_f U(p_f, q_f)$. Deligne-Mostow prove that under certain conditions, the projection to the f-th factor is discrete.

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Write $\mu_i = \{\frac{k_i f}{d}\}$ where, for real x, $\{x\}$ denotes the fractional part of x. Write $\mu_\infty = 2 - \sum_i \mu_i$.

Theorem

(Deligne-Mostow) Assume that for all suffices (including ∞), we have $\mu_i + \mu_j < 1$. (1) The unitary group of the hermitian form h on M_f is $U(n-2,1)$ if and only if $0 < \mu_\infty < 1$.

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(2) Assume (1). The projection to the f-th factor of Γ is discrete if for all suffices i, j (including ∞), the condition $\frac{1}{1-\mu_i-\mu_j} \in \mathbb{Z}$ if $miu_i \neq \mu_j$ holds. If $\mu_i = \mu_j$ then this reciprocal is allowed to be half integral.

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(3) If there is another f' such that $U(p_{f'}, q_{f'})$ is non-compact, then the f-th projection is a non-arithmetic lattice in $U(n-2,1)$. Moreover, Γ is thin.

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(3) If there is another f' such that $U(p_{f'}, q_{f'})$ is non-compact, then the f-th projection is a non-arithmetic lattice in $U(n-2,1)$. Moreover, Γ is thin.

For example, take $d = 18, n = 4$ and each $k_i = 1$. Take $f = 7 \in (\mathbb{Z}/18\mathbb{Z})^*$. Then $\mu_i = \frac{7}{18}$, and $\mu_\infty = 2 - 4\frac{7}{18} = \frac{4}{9}$ lies between 0 and 1. Moreover, $\frac{1}{1-\mu_i-\mu_j} = \frac{1}{1-14/18} = \frac{9}{2}$ is a half integer. Then $U(p_7, q_7) \simeq U(2,1)$ and $U(p_5, q_5) \simeq U(2,1)$. Hence the monodromy is thin.

If we are to have rank one factors, then by the above theorem, $\mu_\infty > 0$ that is $\sum\{\frac{k_i f}{d}\} < 2$. Since each $\mu_i \geq \frac{1}{d}$, it follows that $2 \geq n(\frac{1}{d})$ i.e.$n \leq 2d$. Thus, if we take $n \geq 2d + 1$, then none of the factors of the unitary group $U(h)K \otimes \mathbb{R})$ will have real rank one. Therefore, the above criterion is not applicable.

Getting Unipotent Elements

The proof consists in showing that there are many unipotent elements in the mondromy group, in the range $n \geq 2d + 1$. Take $n = 2d$, to begin with. Then the Hermitian form is degenerate (has a one dmensional kernel). The representation ρ_d of B_n is reducible, since n is divisible by d: it has a one dimensional invariant subspace, with the quotient being the representation $\overline{\rho}_d$ and hence the matrices of the monodromy group are of the form $\rho_d(g) = \begin{pmatrix} 1 & ** \\ 0 & \overline{\rho}_d(g) \end{pmatrix}$ for $g \in B_n$.

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We now look at the subgroup $B_{n-1} \subset B_n$, acting only on the first $n-1$ strands. The restriction of $\overline{\rho_d}$ to the smaller group is simply the Burau for the smaller group. Thus the matrices of B_{n-1} are of the form $\begin{pmatrix} 1 & 0 \\ 0 & \rho_d(g) \end{pmatrix}$; it can be shown that there exist central elements c in B_{n-1} whose image in the quotient are scalars $\lambda \neq 1$.

Getting many unipotents

We now consider the commutator

$$[\rho_d(g), \rho_d(c)] = [\begin{pmatrix} 1 & ** \\ 0 & \overline{\rho}_d(g) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}] = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}]$$

for some $* \neq 0$. This is a nontrivial unipotent element in the unipotent radical of the unitary group $U(h)$ (remember that h is degenerate).

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If we now take $n = 2d + 1$, the conclusion of the preceding paragraph says: the image of B_{n-1} under the Burau representation of B_n contains an arithmetic subgroup of the unipotent radical of a parabolic subgroup, namely, the one which preserves the flag

$$\mathbb{C}v \subset v^{\perp} \subset \mathbb{C}^{n-1}.$$

The arithmeticity follows from

A criterion for Arithmeticity

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Suppose that G is an absolutely simple linear algebraic group defined over a number field K; denote by O_K the ring of integers in K. Suppose G is such that

$$\infty - rank(G) = \sum_{v \quad archimedean} K_v - rank(G) \geq 2,$$

and $K - rank(G) \geq 1$. Suppose that $\Gamma \subset G(O_K)$ is a Zariski dense subgroup in G, such that the intersection of Γ with the integer points $U(O_K)$ has finite index in $U(O_K)$ where U is the unipotent radical of a maximal parabolic subgroup of G defined over K. Then (M.S.Raghunathan, Pacific Journal of math, 152 (1992), and T.N.Venkataramana, Systems of Generators, Pacific Journal of math, 166 (1994))) Γ is arithmetic, i.e. Γ has finite index in $G(O_K)$.

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This immediately implies that the image of the Burau representation ρ_d is arithmetic provided $n = 2d + 1$. Then a "bootstrapping" plus induction gives the same result for all $n \geq 2d + 1$.

Thank you for your attention.