From graphs to free probability

joint work with

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1. Abstract

We investigate a construction which associates a finite von Neumann algebra $M(\Gamma, \mu)$ to a finite weighted graph (Γ, μ). This also yields a 'natural' example of a Fock-type model of an operator with a free Poisson distribution.

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In general, the algebra $M(\Gamma, \mu)$ is a free product, with amalgamation over a finite-dimensional abelian subalgebra corresponding to the vertex set, over algebras associated to subgraphs 'with one edge' (actually a pair of dual edges).

1. **von Neumann algebra :** A *-closed subalgebra M of $B(\mathcal{H})$ satisfying: (a) M is closed under strong ($=$ pointwise) convergence of sequences - i.e., if $x_n \in M, x_n\xi \to x\xi \ \forall \xi \in \mathcal{H} \Rightarrow x \in M$, or equivalently, (b) $M = (M')'$.

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2. A functional τ on an algebra A is said to be
   - a **trace** if it satisfies $\tau(xy) = \tau(yx) \ \forall x, y \in A$.
   - **positive** if A is a *-algebra and $\tau(x^*x) \geq 0 \ \forall x \in A$.
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4. A **finite von Neumann algebra in standard form** is a von Neumann algebra $M \subset B(\mathcal{H})$ which admits a cyclic unit vector ξ such that the equation $\tau(x) = \langle x\xi, \xi \rangle$ defines a faithful tracial state on M.

5. Slide 6 will use terms from Voiculescu's **free probability theory**.

By a weighted graph we mean a tuple $\Gamma = (V, E, \mu)$, where:

- V is a (finite) set of vertices;
- E is a (finite) set of edges, equipped with 'source' and 'range' maps $s, r : E \to V$ and '(orientation) reversal' invoution map $E \ni e \mapsto \tilde{e} \in E$ with $(s(e), r(e)) = (r(\tilde{e}), s(\tilde{e}))$; and
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Let $P_n(\Gamma)$ denote the vector space with basis $\mathcal{P}_n(\Gamma) = \{[\xi] : \xi$ is a path of length n in $\Gamma\}$. We think of a path $\xi = \xi_1 \xi_2 \cdots \xi_n$ as the 'concatenation product' where ξ_i denotes the i-th edge of ξ.

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We write $F\Gamma = \oplus_{n \geq 0} P_n(\Gamma)$ for the indicated direct sum, and equip it with this multiplication: if $[\xi] \in \mathcal{P}_m(\Gamma), [\eta] \in \mathcal{P}_n(\Gamma)$, then $[\xi]\#[\eta] = \sum_{k=0}^{\min(m,n)} [\zeta_k]$,

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where $[\zeta_k] \in \mathcal{P}_{m+n-2k}$ is defined by

$$[\zeta_k] = \begin{cases} \frac{\mu(r(\xi_m))}{\mu(r(\xi_{m-k}))} [\xi_1 \xi_2 \cdots \xi_{m-k} \eta_{k+1} \eta_{k+2} \cdots \eta_n] & \text{if } \xi_{m-j+1} = \widetilde{\eta_j} \forall 1 \leq j \leq k \\ 0 & \text{otherwise} \end{cases}$$

In particular, notice that $\mathcal{P}_0(\Gamma) = \{[v] : v \in V\}$, and that if $v = s(\xi), w = r(\xi)$ for some $[\xi] \in \mathcal{P}_n$, and if $u_1, u_2 \in V$, then $[u_1][\xi][u_2] = \delta_{u_1,v}\delta_{u_2,w}[\xi]$; and less trivially, if $[\xi] \in \mathcal{P}_1$ and $[\eta] \in \mathcal{P}_m$, $m \geq 1$, then

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We define $\phi : F\Gamma \to P_0$ by requiring that if $[\xi] \in P_n$, then

$$\phi([\xi]) = \begin{cases} 0 & \text{if } n > 0 \\ [\xi] & \text{if } n = 0 \end{cases}$$

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and finally define

$$\tau = \mu^2 \circ \phi$$

where we simply write μ^2 for the linear functional on $P_0(\Gamma)$ which agrees with μ^2 on the basis $\mathcal{P}_0(\Gamma)$.

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The definitions show that for $[\xi], [\eta] \in \cup_n \mathcal{P}_n(\Gamma)$, we have

$$\tau([\xi][\eta]^*) = \delta_{\xi,\eta}\mu(r(\xi))\mu(s(\xi)) ,$$

and hence, if we write $(\xi) = (\mu(s(\xi))\mu(r(\xi)))^{-\frac{1}{2}}[\xi]$, then:

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$\{(\xi) : [\xi] \in \cup_{n \geq 0} \mathcal{P}_n(\Gamma)\}$ is an orthonormal basis for $\mathcal{H}(\Gamma) = L^2(F\Gamma, \tau)$.

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1. Suppose $|V| = |E| = 1$, say $V = \{v\}$ and $E = \{e\}$. Then we must have $e = \tilde{e}, s(e) = r(e) = v, \mu(v) = 1, \mathcal{P}_n = \{[e^n]\}$ and $\{\xi_n = (e^n) : n \geq 0\}$ (where $e^n = ee \overset{n\ terms}{\cdots} e$ and $e^0 = v$) is an orthonormal basis for $\mathcal{H}(\Gamma)$; and the definitions show that $x = \lambda(e)$ satisfies $x\xi_n = \xi_{n+1} + \xi_{n-1}$. Thus x is a **standard semi-circular element** and $M(\Gamma) = \{x\}'' \cong L\mathbb{Z}$.

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2. Suppose $|V| = 1, |E| = 2$, say $V = \{v\}$ and $E = \{e_1, e_2\}$ suppose $e_2 = \widetilde{e_1}$. Then we must have $s(e_j) = r(e_j) = v, \mu(v) = 1$. Further $\{[e_1], [e_2]\}$ is an orthonormal basis for $\mathcal{H}_2 = P_1(\Gamma)$, and $P_n(\Gamma)$ is isomorphic to $\otimes^n \mathcal{H}_2$. Thus $\mathcal{H}(\Gamma)$ may be identified with the full Fock space $\mathcal{F}(\mathcal{H}_2)$ and the definitions show that $x_1 = \lambda([e_1])$ may be identified as $x_1 = l_1 + l_2^*$, where the l_j denote the standard creation operators. It follows that x_1 is a **circular element** and $M(\Gamma) = \{x_1\}'' \cong LF_2$.

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3. Suppose $|V| = 2, |E| = 2$, say $V = \{v, w\}$ and $E = \{e, \widetilde{e}\}$ with $s(e) = v, r(e) = w$ and $\mu(w) \leq \mu(v)$. Write $\rho = \frac{\mu(v)}{\mu(w)} (\geq 1)$. If we let $p_v = \lambda([v]), p_w = \lambda([w])$, it follows that $\mathcal{H}_v = ran\ p_v$ has an orthonormal basis given by $\{(\eta_n) : n \geq 0\}$, with $(\eta_n) = [e\tilde{e}e\tilde{e} \cdots (n\ terms)] \in \mathcal{P}_n$ while $\mathcal{H}_w = ran\ p_w$ has orthonormal basis $\{(\xi_n) : n \geq 0\}$ where $(\xi_n) = [\tilde{e}e\tilde{e}e \cdots (n\ terms)] \in \mathcal{P}_n$.

By definition, $M(\Gamma) = \{x\}''$ where $x = \lambda([e])$. Note that the operator x has a matrix decomposition of the form

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$$\begin{aligned} t[\xi_n] &= x[\xi_n] \\ &= [e]\#[\tilde{e}e\tilde{e}e\cdots(n\ terms)] \\ &= [\eta_{n+1}] + \rho^{-1}[\eta_{n-1}]\ ; \end{aligned}$$

and hence,

$$\begin{aligned} t(\xi_n) &= (\mu(s(\xi_n)\mu(r(\xi_n))^{-\frac{1}{2}} t[\xi_n] \\ &= (\mu(w)\mu(r(\xi_n))^{-\frac{1}{2}} \left([\eta_{n+1}] + \rho^{-1}[\eta_{n-1}]\right) \\ &= (\rho^{-1}\mu(v)\mu(r(\eta_{n\pm1}))^{-\frac{1}{2}} \left([\eta_{n+1}] + \rho^{-1}[\eta_{n-1}]\right) \\ &= \rho^{\frac{1}{2}}(\eta_{n+1}) + \rho^{-\frac{1}{2}}(\eta_{n-1}) \end{aligned}$$

It turns out that t^*t has has absolutely continuous spectrum. The proof of the italicised statement involves some nice operator theory.

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Since $p_v + p_w = 1$ and $z \leq p_v$, the definitions are seen to show that $M(\Gamma, \mu)$ is isomorphic to $\mathbb{C} \oplus M_2(L\mathbb{Z})$[2] via the unique isomorphism which maps p_v, p_w, z, u and $|t|$, respectively, to

$(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}), (0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}), (1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}), (0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$, and $(1, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix})$ for some positive a with absolutely continuous spectrum which generates $L\mathbb{Z}$ as a von Neumann algebra.

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4. the proof is completed by an analysis of the Cauchy transform of the scalar spectral measure of such a perturbation, and using Stieltje's inversion formula; which analysis actually shows that this operator has a **free Poisson distribution** in the vector state given by δ_0.

For each dual pair $e, \tilde{e}$ of edges, write $E_e = \{e, \tilde{e}\}$, so $|E_e| \leq 2$. We shall write $\Gamma_e = (V, E_e, \mu)$ with source, range and reversal in E_e as in E. We have the following theorem, whose proof we do not have the time to go into here.

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With the foregoing notation, we have:

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It follows from this theorem and our Example 2 that no matter how the 'orientation reversal map' $e \mapsto \tilde{e}$ is defined on a graph Γ with 1 vertex and n edges, then

$$M(\Gamma, \mu) \equiv LF_n$$

for the unique weighting μ on Γ.

11. References

GJS1 Alice Guionnet, Vaughan Jones and Dmitri Shlyakhtenko, (arxiv: 18/12/07; Conf in honour of Alain Connes, 2010)

KS (IJM, 2009), Kodiyalam, Sunder

JSW (arXiv 2008) Jones, Shlyakhtenko, Walker

KS1 *On GJS for Kac algebras*, (JFA, 2009)

KS2 *On GJS for Graphs*, (JFA, 2011)

GJS2 (JFA, 2011)

BKS Madhusree Basu, Vijay Kodiyalam and V.S. Sunder, *From graphs to free products*, Proc. (MathSci) of the Indian Acad. of Sciences, Vol. 122, No. 4, November 2012, 547-560, and e-print arXiv math. OA/1102.4413.

12. From the first section in [BKS]

There has been a serendipitous convergence of investigations being carried out independently by us (Vijay and me) on the one hand, and by Guionnet, Jones and Shlyakhtenko on the other - see [GJS1], [KS1], [KS2], [GJS2]. As it has turned out, we have been providing independent proofs, from slightly different viewpoints, of the same facts. Both the papers [KS2] and [GJS2], establish that a certain von Neumann algebra associated to a graph is a free product with amalgamation of a family of von Neumann algebras corresponding to simpler graphs. The amalgamated product involved subgraphs indexed by vertices in [KS2], while the subgraphs are indexed by edges in [GJS2]. This paper was motivated by trying to understand how the proof of our result in [KS2] was also drastically simplfied by considering edges rather than vertices.

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Case for increasing operator algebra content:

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