Integral representations for L-functions and Hecke operators

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In this talk, we will describe a relatively elementary approach to the second problem.

The first example

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We have associated the coefficient $\alpha_p = 1$ of p^{-s} to each prime number p. It may be viewed as element in the torus $\mathrm{GL}_1(\mathbb{Q}_p)$.

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The function $\Delta(z)$ is an example of a cusp form (or more generally, a modular form) of weight 12.

Modular forms and Hecke operators

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$$f(\gamma \cdot z) = (cz + d)^k f(z)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ and which satisfies a growth condition at the cusp infinity of $\Gamma\backslash\mathbb{H}$.

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If $a_0 = 0$, we say that f is a cusp form.

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We can define a natural family (actually an algebra) of commuting self-adjoint operators T_m, $(m, n) = 1$ which act on this space. When $m = p$, a prime the action is given by

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If f is a simultaneous eigenform for all the Hecke operators, and if $T_m \cdot f = m^{1-\frac{k}{2}} \lambda_m(f)$, gives the eigenvalue, then it turns out that the λ_m are multiplicative.

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Takeaway: To f we can associate a pair of complex numbers $(\alpha_p, \alpha_p^{-1})$ for each prime number p. This pair can be thought of as an element of the torus in $\mathrm{GL}_2(\mathbb{C})$.

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Takeaway: To f we can associate a pair of complex numbers $(\alpha_p, \alpha_p^{-1})$ for each prime number p. This pair can be thought of as an element of the torus in $\mathrm{GL}_2(\mathbb{C})$.

If f is a cusp form, the L-series is entire and satisfies a functional equation

$$L(s,\pi_f) = (2\pi)^{-(s+\frac{k-1}{2})}\Gamma\left(s+\frac{k-1}{2}\right)D(s,\pi_f) = i^k L(1-s,\pi_f).$$

The local Galois L-function

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$$\pi_v \rightsquigarrow \sigma_{F_v}(\pi_v),$$

where $\sigma_{F_v}(\pi_v)$ is an n-dimensional complex representation of W_{F_v}. For each representation r of $\mathrm{GL}_n(\mathbb{C})$, we define the local L-function as follows:

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where $\sigma_{F_v}(\pi_v)$ is an n-dimensional complex representation of W_{F_v}. For each representation r of $\mathrm{GL}_n(\mathbb{C})$, we define the local L-function as follows:

$$L(s, \pi_v, r) := L(s, (r \circ \sigma_{F_v}(\pi_v)),$$

where the right hand side is the Galois L-function. It has the form $1/P_0(q_v)^{-s}$ where q_v is the cardinality of the residue field and P_0 is a polynomial with $P_0(0) = 1$.

The standard L-function

An n-dimensional representation of W_{F_v} can be thought of as an n-tuple of compex numbers $\alpha_{1,v}, \ldots, \alpha_{n,v}$. For all but finitely many non-archimedean places v, $\alpha_{i,v} \neq 0$ for all $1 \leq i \leq n$. These are the unramified places and the the $\alpha_{i,v}$ Satake parameters of π.

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When r is the standard representation, we simply write $L(s, \pi_v)$ instead of $L(s, \pi_v, r)$. It has the form

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$$L(s, \pi_v) = \prod_i \Gamma\left(\frac{s + \alpha_{i,v}}{2}\right).$$

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$$L(s, \pi_v, \wedge^2) = \prod_{i,j,i<j} \frac{1}{(1 - \alpha_{i,v}\alpha_{j,v} q_v^{-s})}.$$

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We can write similar formulæ for $L(s, \pi_v, \mathrm{Sym}^2)$.

Global representations and L-functions

Our primary objects of study will be the L-functions of global admissible representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$. One way such representations arise are from cuspidal automorphic representations π of GL_n. For instance, every holomorphic eigenform gives rise to such a representation (for $n = 2$).

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By our remarks above this is simply the product of finitely many Γ-functions with a Dirichlet series.

How to study global L-functions

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The second method is the theory of integral representations. We will need this theory only for the standard L-functions of $\mathrm{GL}_n(\mathbb{A}_F)$. It involves expressing the L-function as a Mellin transform of a suitable function on $\mathrm{GL}_n(\mathbb{A}_F)$.

The hypotheses

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(FE) The L-function satisfies a functional equation of the form

$$L(s, \Pi) = AB^s L(1 - s, \tilde{\Pi}),$$

where $\tilde{\Pi}$ is the representation contragredient to Π, and $B > 0$.

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2. *The locations of the possible poles can be recovered from the Satake parameters at two unramified places.*
3. *If* $\sigma = 1$*, then the poles necessarily lie on the lines* $Re(s) = 0$ *and* $Re(s) = 1$*.*

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3. *If* $\sigma = 1$, *then the poles necessarily lie on the lines* $Re(s) = 0$ *and* $Re(s) = 1$.

Note that the hypotheses have placed no restriction on the number, nature and location of the poles. Only the finiteness of their number has been assumed.

The main theorem

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Note that the hypotheses have placed no restriction on the number, nature and location of the poles. Only the finiteness of their number has been assumed. The crucial point is that the existence of an Euler product and Ramanujan bounds on the average, preclude having poles in the critical strip.

Refinements for $n = 2$ - Converse Theorems

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This is a strengthening of the celebrated converse theorem of Weil-Jacquet-Langlands using an improvement of Piatetski-Shapiro. We no longer require that our L-functions be entire, only that they have at most a finite number of poles. This improves on results of W. Li and Booker-Krishnamurthy in this direction (generalisations for $n > 2$ possible).

The Langlands-Shahidi method

Thanks to the work of Langlands, Shahidi, Kim-Shahidi and Gelbart-Shahidi (and others) we know that method of Langlands-Shahidi yields an L-function $L_{Sh}(s, \pi, r)$ which satisfies the properties (C), (FNP) and (FE) when π is a (unitary) cuspdial automorphic representation and $r = \wedge^2, \mathrm{Sym}^2$. Henniart has shown that the local functions $L_{Sh}(s, \pi_v, r)$ and $L(s, \pi_v, r)$ coincide at all places.

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It follows that the Dirichlet series $D_{JS}(s, \pi, \wedge^2)$ satisfies a functional equation with the factors at infinity given by $L(s, \pi_v, \wedge^2)$, $v \mid \infty$

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Thus the only possible poles of $L(s, \pi, \wedge^2)$ are at 0 and 1 and those that arise from $L(s, \pi_v, \wedge^2)$, with $v \mid \infty$. However the local L-factors at infinity have the form $\Gamma\left(\frac{s+\alpha_{i,v}+\alpha_{j,v}}{2}\right)$, where $\mathrm{Re}(\alpha_{\mathrm{i,v}} + \alpha_{\mathrm{j,v}}) > -1$ using the bounds of Jacquet-Shalika.

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It follows that $D(s, \pi, \wedge^2)$ cannot have poles other than at 1 on the line $\mathrm{Re}(s) = 1$. On the other hand our main theorem says that it cannot have poles anywhere else in the critical strip.

The final theorem for the exterior square L-function

Theorem

Let π be a cuspidal automorphic representation. The L-function $L(s, \pi, \wedge^2)$ is entire unless π is self-dual and ω_π is trivial. In the latter case it will have a simple poles at $s = 0$ and $s = 1$ if and only if there is non-vanishing Shalika period.

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Remarks:

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(2) Both these theorems will follow from the works of Arthur+Moeglin-Waldspurger on the trace formula together with the work of Grbac-Shahidi. We emphasise that our approach is much more elementary with no reliance on the trace formula. It uses only tools and ideas from the previous century.

Further remarks - appeals to a higher authority

(3) Piatetski-Shapiro was a proponent of using the method of integral representations (also known as the Rankin-Selberg method) to analyse L-functions. He believed that the poles of L-functions could be more easily identified by the Rankin-Selberg method. He was fond of observing that "Arthur's method is more general, but this approach is much simpler". Our main theorem is a modest step in this direction.

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(4) In principle, we could try to apply our idea to retrieve older holomorphy results for the L-functions of symmetric powers of GL_2 and this method might simplify some of the proofs of Kim-Shahidi. Some of these L-functions are not part of the trace formula framework.

The main idea

The basic idea behind the proof is the following: The poles of the L-functions can be recovered from the asymptotics of Whittaker functions. The fact that the set of these exponents is forced to remain invariant under the action of suitable Hecke operators, severely restricts the possibilities for this set.

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$$L(s, \pi_f) = \int_0^\infty f(iy) y^{s-\frac{1}{2}} d^*y.$$

The Bochner correspondence

The starting point is the (Riemann-Hecke-)Bochner correspondence. Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ and its L-function $L(s)$,

$$\varphi(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n e^{2\pi i n z},$$

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Using the functional equation, standard arguments involving the inverse Mellin transform and shifting the line of integration using the Phragmén-Lindelöf principle yields an equation of the form

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$$q(z) = \sum_{i=1}^{m} p_i(\log z) z^{\beta_i},$$

where the $p_i(t)$ are polynomial functions.

Asymptotic exponents and poles

The key point is that the term z^{β_i} appears in $q(z)$ if and only if $-\beta_i + 1/2$ is a pole of $L(s)$.

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We have thus related the exponents in the asymptotic expression for $\varphi(z)$ to the poles of the L-function.

Applying the Hecke operators

If we apply the Hecke operator T_p to this situation we obtain a relation between the asymptotics:

$$p^{-1/2}q_1(pz) + p^{1/2}q_1\left(\frac{z}{p}\right) \sim a_p q_1(z).$$

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If we set $X = p^{\beta-1/2}$ we get a quadratic equation in X. Note that we get the same equation at every unramified place. Thus β can be completely determined if we know the Hecke eigenvalue at two unramified places.