On the BRUN-TITCHMARSH Inequality

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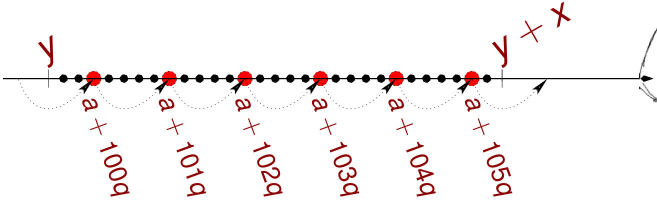
LOOK FOR PRIMES IN:



- Avoid extreme case: $q < x$,
- Seek UPPER BOUNDS,
- We want primes: choose a prime to q,
- No a is special: equidistribution in the $\phi(q)$ classes.

Brun-(Titchmarsh, 1930) Theorem:

Terminology of (Linnik, 1961)

Theorem
(Montgomery & Vaughan, 1974)

When $x > q \geq 1,$

$$\sum_{\substack{y<p\leq y+x,\\ p\equiv a[q]}} 1 \leq \frac{2\,x}{\phi(q)\log(x/q)}$$

$$\pi(10^{10000}+1000)-\pi(10^{10000}) \leq 289.$$

Better than 2? Oups !!!

Often used with a global lower bound, for instance:

Theorem *Let* $1 \le q \le X^{1/10}$. $(X \ge X_0)$
$\forall a$ prime to q,
$\exists m = p_1p_2$ or $p_1p_2p_3 \le X \ / \ m \equiv a[q]$.

Proof. $x = X^{1/3}$.

- $\pi(x) \ge x/\log x$ when $x \ge 17$,
- $\frac{1}{2}\log(x/q) \ge 7/20 > 1/3$,
- Brun-Titchmarsh → one class / three has primes,
- Add. comb. → products of three covers a $b \cdot H$,
- $\exists n$ outside H → result.

Two or three prime factors! See later

The Brun-Titchmarsh Theorem: OPTIMAL or NOT?

- Beats the trivial $1+x/q$ in a wide range,
- When $q = 1$ and $y = 0$,

  the estimate is sharp up to the 2,
- Idem when q is small,

- At size $x + y$, average density is $1/\log(y + x)$
- When $y = 0$, density is $1/\log x$, not $1/\log(x/q)$

What about a heuristics?? Consequences?

Background on Siegel zero

- Study of primes
  $\longrightarrow$ Riemann-ζ function and Dirichlet L-series
- One of the keys: no zero close to $\Re s = 1$
- Hadamard and de la Vallée-Poussin – 1896
  Assume $L(1+i\gamma,\chi)=0$. Take $\sigma>1$:

Bounded if $\chi^2 \neq \chi_0$ or $\gamma \neq 0$

$$3\Re\frac{-L'}{L}(\sigma,\chi_0)+4\Re\frac{-L'}{L}(\sigma+i\gamma,\chi)+\Re\frac{-L'}{L}(\sigma+2i\gamma,\chi^2)\geq 0$$

Pole: $\dfrac{1}{\sigma-1}$

Zero: $\dfrac{-1}{\sigma-1}$

$\Longrightarrow$ $L(1+it,\chi)\neq 0$

Balasubramanian & Ramanchandra, 1976

Problem left: $\chi^2 = \mathbb{1}$ and $\gamma = 0$. A possible exceptional "Siegel zero"

$\chi \neq \mathbb{1}$

More on Balasubramanian & Ramachandra's idea

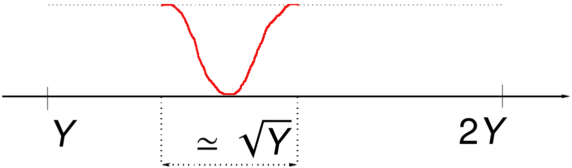
Here is a sample of what they do.

oscillates wildly!

Theorem

$$\sum_{Y<p\leq 2Y}\Bigl(1+\cos(\sqrt{Y}\log p)\Bigr)\gg Y/\log Y.$$

- Analytical means → nothing, *even on RH*,
- Vinogradov → nothing [multiplicativity].



SIEGEL ZERO VERSUS CLASS NUMBER PROBLEM

An algebraical interpretation:

- No Siegel zero

Same as ▸ A lower bound for $L(1,\chi)$ (real analysis, tricks)

Same as ▸ A lower bound for the class number of $\mathbb{Q}(\sqrt{-q})$ (split primes)

Same as ▸ Counting reduced quadratic forms in a family (quadratic system)

↑

Dirichlet class number formula

We are again stuck on these three new points –

No better luck with upper bounds!

BRUN-TITCHMARSH VERSUS SIEGEL ZERO

When $L(1-\delta,\chi) = 0$ and $\delta = o(1/\log q)$:

Cheating a bit

$$\sum_{\substack{p \le X, \\ p \equiv a[q]}} 1 \sim \frac{\pi(X)}{\phi(q)}\Big(1 - \chi(a)\frac{X^{-\delta}}{1-\delta}\Big)$$

(Gallagher, 1970)

when $\dfrac{\log X}{\log q} \to \infty$

For X in some range,
For a such that $\chi(a) = -1$:

$$\sum_{\substack{p \le X, \\ p \equiv a[q]}} 1 \sim \frac{2X}{\phi(q)\log X}$$

Back to the same factor 2 !!

We know the subgroup structure

Brun-Titchmarsh versus Siegel zero

and EFFECTIVITY

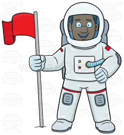
Cheating a bit (Motohashi, 1979), (Basquin, 2006)

- Show that $\sum_{\substack{p \le X,\\ p \equiv a[q]}} 1 \le (2-\varepsilon)\pi(X)/\phi(q)$ for some $\varepsilon > 0$ and all $q \le (\log X)^c$,

Same as
- Prove in an effective fashion that

$$\sum_{\substack{p \le X,\\ p \equiv a[q]}} 1 \sim \pi(X)/\phi(q) \text{ for } q \le (\log X)^c,$$

Same as
- Make $L(1,\chi) \gg 1/q^{1/c}$ effective,



(Goldfeld, 1975), (Gross & Zagier, 1983), (Oesterlé, 1985).

Reinforced Deuring-Heilbronn phenomenom

Brun-Titchmarsh: crossing the second wall!

$1 \geq \alpha > 0$

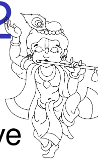
- When $q \leq x^{\alpha}$, $\sum_{\substack{p \leq x, \\ p \equiv a[q]}} 1 \leq \frac{2c(\alpha)x}{\phi(q)\log(x/q)}$ with $c(\alpha) < 1$, (Motohashi, 1974), (Iwaniec, 1982), (Friedlander & Iwaniec, 1997)
- (Maynard, 2013) has $c(\alpha)(1-\alpha) = 1$ when $\alpha \leq 1/8$,
- We always have $c(\alpha) \log x \geq \log(x/q)$.

A FOOTNOTE FOR SPECIALISTS

Brun-Titchmarsh inequality $\leftrightarrow$ zero-free region

Large sieve extension of Brun-Titchmarsh inequality $\leftrightarrow$ *Log-free* density estimates

A SECOND OCCURRENCE OF THE FACTOR 2



(Selberg, 1949) developped by (Bombieri, 1976)

Take an optimal linear sieve

$\omega(n)$ = number of prime factors of n

Contribution of integers with EVEN $\omega(n)$

Expected $=$

Contribution of integers with ODD $\omega(n)$

$+$

$=$ ←*by the sieve*

TWICE the expectation

- We had an analytical hurdle
- This one is a combinatorial hurdle

A FIRST STOP!

WHERE DID THE INTERVALS DISAPPEAR?

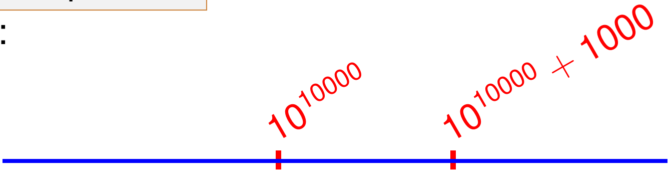
- In q-aspect: the factor 2 is NOT justified.
- Is this factor required by the interval aspect?

We consider $q = 1$.

Towards a lower bound: Prime packing

Locating the problem

Typically :

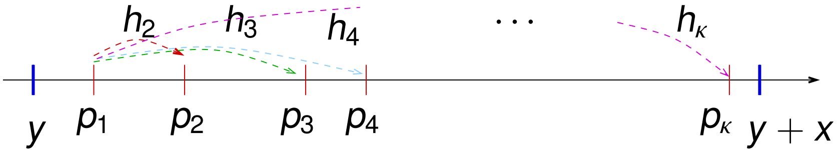


how many primes there?

In $(y, y + x]$, with x **VERY** small,
how many primes At Most?

On average $x/log(y + x)$

Let us look at the spacings:



$$p_1 + (h_1 = 0, h_2, h_3, \cdots, h_\kappa)$$

A κ-tuple is *admissible* $(h_1, h_2, \cdots, h_\kappa)$ when

$$\forall q \geq 1, \quad \{h_1, h_2, \cdots, h_\kappa\}/q\mathbb{Z} \neq \mathbb{Z}/q\mathbb{Z}$$

Enough: q prime and $q \leq \kappa^\star$.
$(2, 4, 6)$ no!

Framework of ... (Hardy & Littlewood, 1922)!

Length of a κ-tuple : $L(h) = h_\kappa - h_1 + 1$

PROBLEM: Given κ, find h that minimizes $L(h)$.

Theorem (Hensley & Richards, 1974)

Let $\varepsilon > 0$ and L be large.
$\exists$ admissible κ-tuple of length $\leq L$ such that

$$\kappa \geq \pi(L) + (\log 2 - \varepsilon)\frac{L}{(\log L)^2}.$$

Schinzel: $\log 2 \to 2\log 2$ *most probably*

A numerical approach

Theorem (Dusart, 1998)

$\exists$admissible 1415-tuple of length $L = 11763$

$$\pi(11763) = 1409 \text{ , hence } +6 \text{ !! } \quad \kappa \geq 1.004\pi(L)$$

$(2\pi(11763/2) = 1550 > 1415)$

PROBLEM: $\max_{h \text{ admissible}} \frac{\kappa}{\pi(L)}$? ≥ 1.004

PROBLEM: $\limsup_{\kappa\to\infty} \max_{h \text{ admissible}} \frac{\kappa}{\pi(L)}$? $\in [1,2]$

A MORE MODEST PROJECT

Back to diminishing the upper bound

Statement and history

Theorem (O.R. & S. Yazdani, 2016 ?)

When $x \geq x_0$, $\pi(y+x) - \pi(y) \leq \dfrac{2x}{5.66 + \log x}$

(Bombieri, 1971) $5.66 \to -3$

(Montgomery & Vaughan, 1973) $5.66 \to 5/6$

(Selberg, 1991) $5.66 \to 2.81$

(O.R. & Schlage-Puchta, 2006) $5.66 \to 3.53$

SIEVING OUT THE SMALL PRIMES MORE PRECISELY

$$\mathfrak{f} = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$$

Cheating a bit!

A MAXIMAL ESTIMATE

$$D_{\mathfrak{f}}(u) = \max_{\substack{h \in \mathbb{R}, \\ w/[w]=[u]}} \left| \sum_{\substack{h<n\le h+w, \\ \gcd(n,\mathfrak{f})=1}} 1 - \text{Model}(w) \right|$$

- Selberg: Model$(w) = \dfrac{\varphi(\mathfrak{f})}{\mathfrak{f}} w$
- Now: Model(w) is optimal
  + expressed as lin. comb. of $\displaystyle\sum_{\substack{n \le w, \\ \gcd(n,\mathfrak{f})=1}} \chi(n)$

where χ are Generalized Dirichlet characters.

Idem for $D_{\mathfrak{f}}(u)$

SOME SIMPLE TOOLS:

$$\mu(q) = \begin{cases} (-1)^r & \text{when } q = p_1 \cdots p_r,\ p_i \neq p_j, \\ 0 & \text{else.} \end{cases},$$

$$\phi(q) = \sum_{\substack{n \le q, \\ (n,q)=1}} 1 \stackrel{\sim}{\approx} q$$

$$\sigma(q) = \sum_{d|q} d \stackrel{\sim}{\approx} q$$

$$c_q(n) = \sum_{\substack{1 \le a \le q, \\ (a,q)=1}} e^{2i\pi an/q}$$ ← *Ramanujan sum*

Main property we use:

$$c_q(n) = \mu(q) \text{ when } (n,q) = \gcd(n,q) = 1$$

SIEVING THE LARGER PRIMES

- Wanted: $f = \mathbb{1}_{n \in (y, y+x] \cap \mathcal{P}}$
- From: Set $= \{n \in (y, y+x], \gcd(n, \mathfrak{f}) = 1\}$ $(y \geq x)$
- Using: $n \in$ Set and prime $\Rightarrow \mu(q)c_q(n) = 1, \forall q \leq \sqrt{x}$

$$\mathfrak{F}(n) = \frac{1}{C} \sum_{\substack{q/\sigma(q) \leq \sqrt{x}, \\ \gcd(q, \mathfrak{f}) = 1}} \left(1 - \frac{\sigma(q)}{\sqrt{x}}\right) \frac{\mu(q)c_q(n)}{\phi(q)}$$

Local-Global gluing + optimization prayer

Choice of $C \longrightarrow \mathfrak{F}(p) = 1$ when $p \in (y, y+x]$

The local-model or functional approach

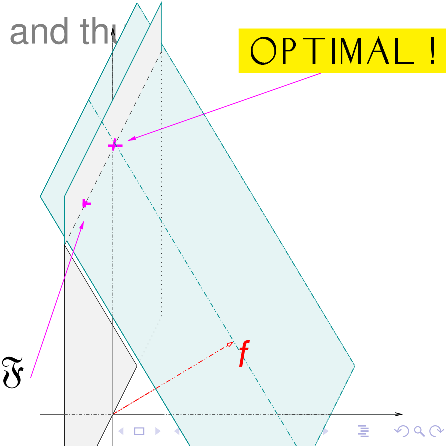
Consider

$$[g_1, g_2] = \sum_{\substack{y < n \leq y+x, \\ (n,\mathfrak{f})=1}} g_1(n)\overline{g_2(n)}$$

We have $[f, f] = [f, \mathfrak{F}] = Z$ and th

$$[\mathfrak{F}, \mathfrak{F}] \geq [f, f]$$

But $[\mathfrak{F}, \mathfrak{F}]$ is computable!



A METHOLOGICAL INTERLUDE

The "small sieve" approach: $|\mathfrak{F}(n)|^2 \geq f(n)$

and thus....

$[\mathfrak{F}, \mathfrak{F}] \geq [f, f]$!! Yet again!!

Local Models → Large sieve $[\mathfrak{F},\mathfrak{F}] = \sum_n \left| \sum_{q \leq \sqrt{x}} \sum_{a \bmod {}^*q} h(a/q) e(na/q) \right|^2$

Local Models → Small sieve $\mathfrak{F}(n) = \sum_{d|n} \lambda_d, \quad \lambda_d = \sum_{d|q\cdots} \cdots$

.... and we work ...

- *Special sieve with a thicker close-to-diagonal,*
- *Generalized characters,*
- *Mellin (and reverse!) transform in several complex variables,*
- *Distribution measure of arithmetical function*

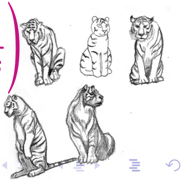
Some funny beasts to study

Cheating a bit

$$A_{\mathfrak{f}}(s) = \prod_{\gcd(p,\mathfrak{f})=1} \left(\frac{p-1}{p}\right)^2 \left(1 + \frac{2}{p}\left(\frac{p+1}{p}\right)^s\right)$$

$$B_{\mathfrak{f}}(s) = \prod_{\gcd(p,\mathfrak{f})=1} \left(1 + \frac{(p+1)^{2s}}{p^4}\right)$$

$$C_{\mathfrak{f}}(s) = \prod_{\gcd(p,\mathfrak{f})=1} \left(1 + \frac{(p+1)^{2s}}{p^{s+2}}\right)\left(1 - \frac{1}{p^{2-s}}\right)$$



Main hurdle left

$$\mathfrak{f} \mapsto \frac{1}{2i\pi}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{A_{\mathfrak{f}}(s)B_{\mathfrak{f}}(s)C_{\mathfrak{f}}(s)\,\zeta(2-s)\,\mathcal{E}(s)ds}{4^s(2-s)^2(1-s)^2 s}$$

$$\mathcal{E}(s) = \sum_{n\geq 1} \frac{D_{\mathfrak{f}}(n) - D_{\mathfrak{f}}(n-1)}{n^s}$$

Even if $\mathcal{E}(s)$ is given, computing is difficult!

First step:

- $A_{\mathfrak{f}}(s)$, $B_{\mathfrak{f}}(s)$ and $C_{\mathfrak{f}}(s)$ $\longrightarrow$ finite Euler products

$(\Delta = \prod_{p \le P,\ p \nmid \mathfrak{f}} p)$, $\qquad \mathcal{G}_0(u) = \frac{1}{2}u^2(\frac{5}{2} - \log u) - u(1 + \log u) - 1/4$

To be evaluated:

$$\frac{\phi(\Delta)^2}{\Delta^2} \sum_{\substack{abc|\Delta, \\ \sigma(a)\sigma(bc)^2 \le wac}} \frac{c\phi_2(c)}{\phi(bc)^2\sigma(bc)^2} \frac{2^{\omega(a)}a}{\phi(a)\sigma(a)} \mathcal{G}_0\left(\frac{\sigma(a)\sigma(bc)^2}{wac}\right)$$

Second step:

- Restrict *a* to having less than 7 prime factors

Output:

- $\mathfrak{f} = 2 \times 3 \times 5 \times 7 \longrightarrow \text{Constant} \ge 4.51$
- $\mathfrak{f} = 2 \times 3 \times 5 \times 7 \times 11 \longrightarrow \text{Constant} \ge 4.91$
- $\mathfrak{f} = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \longrightarrow \text{Constant} \ge 5.38$
- $\mathfrak{f} = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \longrightarrow \text{Constant} \ge 5.66$

Prospectives

Is the 2 required?

I don't know! Hensley & Richards propose: no.

I believe:

$$\exists f(X) \to \infty / \ \pi(y+x) - \pi(y) \leq \frac{2x}{f(x) + \log x}$$

What about the L^q-problem?

$$\int_Y^{2Y} \Big| \sum_{y<p\leq y+x} 1 \Big|^q dy \ll \cdots \ ?$$

Can we really adapt that to primes in progressions?

A FOOTNOTE

(Odlyzko et al., 1999)
give a heuristics that says that:

The most frequent difference is first 6,
then 30,
then 210,
and so on...

They sustain their point by producing calculations.
Marek Wolf started this business by exhibing some surprising numerical tables.

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