Groups with expansive automorphisms

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- In general, neither U_α nor $U_{\alpha^{-1}}$ is closed;
- U_α is closed if and only if $U_{\alpha^{-1}}$ is closed ([BaWi-04]);
- If G is a p-adic Lie group, U_α is closed, in fact an unipotent algebraic group ([Wa-84]).

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Then α is an expansive automorphism but G does not admit any contractive automorphism as the commutator of G is discrete.

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Theorem [Wi-94]

There is a compact open subgroup V such that V_{++} and V_{--} are closed and $V = V_+V_-$: such a subgroup is called a tidy subgroup for α.

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Proposition (GIR)

α is expansive if and only if α restricted to M_α is expansive.

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Here α is never contractive since U_α as well as $U_{\alpha^{-1}}$ is a proper subgroup.

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- We choose a series that has maximum such j, hence subfactors of such a series satisfy (1).
- For each such j we introduce $(G_{j-1} \supseteq) M_j \supseteq N_j (\supseteq G_j)$ so that the conclusion are valid for the subfactors.

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Proof

We restrict to the Levi factor and prove the expansiveness of the factor automorphism.

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Abelian expansive groups

The following are abelian groups with expansive automorphisms:

(1) $\mathbb{Q}_p^n$ for some $n \in \mathbb{N}$, together with a linear automorphism $\beta\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ such that β or β^{-1} is contracting.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let C_p be the cyclic group of order p and $C_p^{(-\mathbb{N})}$ be the restricted direct product.

(2) $C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ with the right-shift;

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(4) $C_p^{\mathbb{Z}}$ with the right-shift.

Theorem (GIR)

Let A be an abelian, totally disconnected, locally compact group and $\alpha\colon A \to A$ be an expansive automorphism. Assume that $A = U_\alpha U_{\alpha^{-1}}$ and every α-stable proper closed subgroup of A is discrete. Then there exists a prime number p such that (A, α) isomorphic to one of the above.

Lie groups over local fields

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An automorphism α of a Lie group over a local field is expansive if and only if the differential $d\alpha$ has no eigenvalue of absolute value one.

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Corollary (GIR)

If a Lie group over a local field has an expansive automorphism, then its Lie algebra is nilpotent.

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Even for a p-adic linear group, U_α may not normalize $U_{\alpha^{-1}}$: recall that U_α as well as $U_{\alpha^{-1}}$ both are closed.

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$$\alpha(x, y, z + \ Z_p) = (x/p, py, z + \mathbb{Z}_p).$$

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Here, $U_\alpha = \{(0, y, \mathbb{Z}_p) \mid y \in \mathbb{Q}_p\}$ and
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Thus, $U_\alpha U_{\alpha^{-1}} = \{(x, y, \mathbb{Z}_p) \mid x, y \in \mathbb{Q}_p\}$ which is not even a group.

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Define $\beta \colon H \to H$ by $\beta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & px & z/p \\ 0 & 1 & y/p^2 \\ 0 & 0 & 1 \end{pmatrix}$

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$U_{\alpha^{-1}} = \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{Q}_p \} \times \{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \}$. In this case $U_\alpha U_{\alpha^{-1}} = G$ but neither U_α nor $U_{\alpha^{-1}}$ normalize the other.

Thanks for your attention!!!