Singularities, Algebraic entropy and Integrability of discrete Systems

K.M. Tamizhmani

Pondicherry University,
India.

Indo-French Program for Mathematics
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Abstract

In this talk, we discuss about two integrability detectors, namely, singularity confinement approach and algebraic entropy method for discrete systems. The power of these methods are demonstrated by deriving discrete Painlevé equations and linearizable systems.

Painlevé developed α-method to test the occurrence of branch point solution and to prove single-valuedness. found the six Painlevé equations whose general solutions require the introduction of new transcendent called Painlevé transcendent.

This classification of second-order equations was completed by Gambier who presented the complete list of fifty equations including the six Painlevé equations that satisfied the requirement of absence of movable critical singularities.

Forty four of them can be solved by known functions. It is clear from the analysis of the above prominent personalities that the integrability of differential equation is related to the existence of the general solution which is single-valued and analytic.

Painlevé property

A system of ODE is said to have the Painlevé property if its general solution has no movable critical singular point.

In other words, an ODE is said to possess the Painlevé property if all movable singularities of all solutions are poles.

For example, the solution of

$$w' + w^2 = 0,$$

is $w = (z - z_0)^{-1}$. Note that, if the initial condition is $w(0) = 1$, then $z_0 = -1$. On the other hand, if the initial condition is $w(0) = 2$, then z_0 moves to $z_0 = -\frac{1}{2}$. That means that the location of the singularity at z_0 moves with initial condition. Such singularities are called movable singularities. For Painlevé property to hold the only forbidden singularities are the movable critical (multi-valued) singularities.

Consider the ODE

$$w'' = w'^2 \frac{2w - 1}{w^2 + 1}.$$

The singularities of this equation are $w = \pm i$, $w = \infty$ and $w' = \infty$. Series expansion can be developed for solutions exhibiting each of the above singular behaviors and the equations passes the Painlevé test. This equation, however, has the general solution

$$w = \tan\{\log[k(z - z_0)]\},$$

where k and z_0 are constants. For $k \neq 0$, w has poles at

$$z = z_0 + k^{-1} exp\{-(n + \frac{1}{2})\pi\},$$

for every integer n. These poles accumulate at the movable point z_0, giving raise to a movable branched nonisolated essential singularity. This example clearly shows that passing the Painlevé test need not guarantee that the equation actually possess the Painlevé property.

Continuous Painlevé equations

$$\mathbb{P}_{\mathrm{I}} \quad w'' = 6w^2 + z$$

$$\mathbb{P}_{\mathrm{II}} \quad w'' = 2w^3 + zw + a$$

$$\mathbb{P}_{\mathrm{III}} \quad w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(aw^2 + b) + cw^3 + \frac{d}{w}$$

$$\mathbb{P}_{\mathrm{IV}} \quad w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - a)w - \frac{b^2}{2w}$$

$$\mathbb{P}_{\mathrm{V}} \quad w'' = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{z} + \frac{(w-1)^2}{2z^2}(aw + \frac{b}{w}) + \frac{cw}{z} + \frac{dw(w+1)}{(w-1)}$$

$$\mathbb{P}_{\mathrm{VI}} \quad w'' = \frac{w'^2}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) - w'\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)$$
$$+ \frac{w(w-1)(w-z)}{2z^2(z-1)^2}\left(a - b\frac{z}{w^2} + c\frac{z-1}{(w-1)^2} - (d-1)\frac{z(z-1)}{(w-z)^2}\right)$$

where a, b, c, d are arbitrary constants.

All the solutions of the Painlevé equations are meromorphic functions and are called Painlevé transcendent. Details see

Painlevé equations have many wonderful properties: By coalescence one can obtain all other Painlevé equations from $\mathbb{P}_{\mathrm{VI}}$ by taking special limits on the parameters; the special function solutions exist for specific values of parameters etc..

Recently, it has been found that Painlevé equations admit Lax pairs, bilinear forms as well. Theory of space of initial conditions was used extensively by to find the Hamiltonian structure of the Painlevé equations by the introduction of tau functions

Einstein's quote

“ To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space".

Numerics

computer simulation of physical phenomena:

based on discretisation

Assumption: (close) parallel between continuous and discrete system

Valid only for small discretisation step(?)

Integrability of discrete systems

Example

Riccati equation

$$x' = \alpha x^2 + \beta x + \gamma$$

Discretisation as a two-point mapping:

$x(t) = x_n$ and $x(t + \Delta t) = x_{n+1}$

Discretisation

$x' \to (x_{n+1} - x_n)/\Delta t$
But x^2?

Possibilities $x^2 \to x_n^2, x_{n+1}^2, x_n x_{n+1},$

$(x_n + x_{n+1})x_n/2, (x_n + x_{n+1})^2/4, (x_n^2 + x_{n+1}^2)/2$

Integrable discretisation $x^2 \to x_n x_{n+1}$

$\to$ Homographic mapping:

$$x_{n+1} = \frac{bx_n + c}{1 + ax_n}$$

Linearizable through a Cole-Hopf transformation

Painlevé P_{I} equation

$$x'' = 3x^2 + t$$

Discretisation (explicit three-point mapping)
$x'' \to (x_{n+1} - 2x_n + x_{n-1})/(\Delta t)^2$
and $3x^2 \to x_n(x_{n+1} + x_n + x_{n+1})$
$\to$ Integrable equation:

$$x_{n+1} - 2x_n + x_{n+1} = \frac{3x_n^2 + z}{1 - x_n}$$

Integrable integrators

What are the discrete Painlevé equations?

Integrable, nonautonomous, discrete

Continuous limit: Painlevé equations (name of d-$\mathbb{P}$'s: second-order systems) *Why are d-$\mathbb{P}$'s (and their study) interesting?*

- Discrete systems: most fundamental objects
- Discrete $\rightarrow$ Continuous through simple limit

(but continuous better known)
Study of discrete: perfect parallel between continuous and discrete integrable systems

- d-$\mathbb{P}$'s: integrable discretisations of c-$\mathbb{P}$'s

But few numerical studies to date

The (incomplete) history of d-$\mathbb{P}$'s

orthogonal polynomials

integrable discrete nonautonomous systems

Hhigher-order d-$\mathbb{P}$'s

(no continuous limits computed)

They predate the continuous-$\mathbb{P}$'s

Much (much) later

orthogonal polynomials

$$x_{n+1} + x_{n-1} + x_n = \frac{z_n}{x_n} + 1$$

with $z_n = \alpha n + \beta$

(many years later was recognised as d-$\mathrm{P_I}$)

contiguity relations of c-$\mathbb{P}$'s

From P_{II}:

$$w'' = 2w^3 + tw + \alpha$$

contiguity relation:

$$\frac{\alpha_n + 1/2}{x_{n+1} + x_n} + \frac{\alpha_n - 1/2}{x_n + x_{n-1}} = -(2x_n^2 + t)$$

where $x_n = w(t, \alpha_n)$ and $\alpha_n = n + \alpha_0$.

No continuous limit was derived!

Same period, integrable difference recursion relations in field theoretical models

No relation to (possibly higher) d-$\mathbb{P}$'s established

Field-theoretical model of 2-D gravity

Recursion relation of Shohat

Computed the continuous limit!

Obtained $w'' = 6w^2 + t$, i.e. Painlevé I

The discrete $\mathbb{P}$'s were born!

Obtained

$$x_{n+1} + x_{n-1} = \frac{z_n x_n}{1 - x_n^2}$$

Continuous limit $w'' = 2w^3 + tw$

P_{II} equation for the zero value of parameter

Previously (with **Capel**) derived d-KdV and d-mKdV

Already (in the 70's) derived by **Hirota**

Approach of

Similarity reduction of mKdV $\rightarrow$ P_{II}

Similarity constraint of d-mKdV (nonlinear)

Similarity reduction$\rightarrow$d-P_{II}

same as

Beginning of 90's

Examples of d-$\mathbb{P}$'s

Obtained with 3 of the 4 main methods:

- orthogonal polynomials (spectral methods)
- reductions
- contiguity relations.

The time was ripe for the 4th method

Main idea:

apply an integrability detector

to some postulated functional form

and select the integrable cases

Starting point:

The QRT, 2nd-order, integrable, autonomous mappings

Asymmetric mapping:

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}$$

Symmetric case ($g_i = f_i$):

$$x_{m+1} = \frac{f_1(x_m) - x_{m-1} f_2(x_m)}{f_2(x_m) - x_{m-1} f_3(x_m)}$$

(identification $x_n \to x_{2m}$, $y_n \to x_{2m+1}$)

Parameter counting: 8 for the asymmetric mapping and 5 for the symmetric

Conserved quantity K given by:

$$\begin{aligned}(\alpha_0 + K\alpha_1)x_n^2 y_n^2 + (\beta_0 + K\beta_1)x_n^2 y_n + (\gamma_0 + K\gamma_1)x_n^2 \\ +(\delta_0 + K\delta_1)x_n y_n^2 + (\epsilon_0 + K\epsilon_1)x_n y_n + (\zeta_0 + K\zeta_1)x_n \\ +(\kappa_0 + K\kappa_1)y_n^2 + (\lambda_0 + K\lambda_1)y_n + (\mu_0 + K\mu_1) = 0\end{aligned}$$

For the symmetric case:

$$(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)x_{n+1}x_n(x_{n+1} + x_n) \\ +(\gamma_0 + K\gamma_1)(x_{n+1}^2 + x_n^2) + (\epsilon_0 + K\epsilon_1)x_{n+1}x_n \\ +(\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\mu_0 + K\mu_1) = 0$$

Invariant relation between x_n and y_n:

2-2 correspondence (similarly for x_n, x_{n+1})
Integration of symmetric correspondence:

elliptic functions (Euler)
Integration of the asymmetric case also (2001)

Solution of QRT mapping: sampling of an elliptic function.

Pertinence of QRT

Continuous $\mathbb{P}$'s: non-autonomous extensions of elliptic functions

($\mathbb{P}$'s: same functional forms as the autonomous equations with coeffs depending on the independent variable)

Strategy: Start from QRT

allow coeffs to depend on indep. variable

select the integrable cases (through integrability detector)

The methodology is following

- Etienne Bezout (1779) theorem: The number of intersection of two plain curves is equal to the product of their degrees.
- Complexity- V.I. Arnold (1990): The growth of the degree sequence is measure of the complexity of the map.
- Slow growth - A.P.Veselov (1982)
- Algebraic entropy - G.Falqui and C.M.Viallet (1993)

Remark: The map is written in terms of projective coordinates

By this process the map must be having homogeneous degrees in both the numerator and denominator, then the degree of the map is computed.

If there is no factorization of the polynomial the sequence then $d_n = d^n$ holds. On the other hand, if there is some factorization appear then it is obvious that the degree sequence should satisfies $d_n \leq d^n$. the drop may even be so important that the growth of d_n becomes polynomial and not exponential.

Arnold:

Complexity number of intersection points of fixed curve with the iterates of a second curve

Exponential growth for generic mappings

Polynomial for integrable mappings

"*integrability has an essential correlation with the weak growth of certain characteristics*"

Viallet, Falqui, Hietarinta:

algebraic entropy

Mapping of degree d

$\rightarrow$ n-th iterate: degree d^n, unless there exist simplifications

Integrable mappings: massive simplifications

$\rightarrow$ polynomial degree growth

The measure of the complexity of the discrete dynamical systems is an interesting area of research. In recent years, this topic assumes importance in the study of the integrability of the dynamics of rational maps.

It has been observed that the dynamical complexity and degree of the composed map has close link.

One can define easily the algebraic entropy of a map from the growth of the degrees of its iterates.

Knowing the degree of the first few iterates, generally, it is possible to find a finite recurrence relation between the degrees which could be solved exactly and obtain a closed form expression.

In addition, it is also possible to find the generating function for the degree sequence. Moreover, from the generating function, one can easily find the exact value of the algebraic entropy.

First point to remember is that, we should consider the bi-rational map in the projective space.

This means that the map is written in terms of projective coordinates.

By this process the map must be having homogeneous degree in both the numerator and denominator.

After canceling the common factors between numerator and denominator, the degree of the map is computed.

If bi-rational map is denoted by ϕ, then we can define the sequence d_n of the degrees of the successive iterates ϕ^n of ϕ.

Here, d_n denotes the number of intersection of the n'th image of ϕ generic line with a fixed hyper plane.

Then growth of the degree sequence d_n is a measure of the complexity of ϕ.

If there is no factorization of the polynomial the degree sequence $d_n = d^n$ holds. On the other hand, if there is some factorization appear then it induces a drop of the degree and thus the degree sequence should satisfy $d_n \leq d^n$.

The drop may even be so important that the growth of d_n becomes polynomial and not exponential.

As an application of the method of algebraic entropy, we consider the map

$$x_{n+1} + x_{n-1} = \frac{a}{x_n} + \frac{1}{x_n^2}.$$

Let us introduce the homogeneous coordinates $x_0 = p$, $x_1 = q/r$, where we assume that $deg\ p = 0, deg\ q = 1$ and $deg\ r = 1$. Now, in term of homogeneous coordinates the first few iterates of the mapping can be written as

$$x_2 = \frac{r^2 + aqr - pq^2}{q^2},$$

$$x_3 = \frac{q\mathbf{p}_4}{r(r^2 + aqr - pq^2)^2},$$

$$x_4 = \frac{(r^2 + aqr - pq^2)\mathbf{p}_6}{{\mathbf{p}_4}^2},$$

$$x_5 = \frac{\mathbf{p}_4\mathbf{p}_9}{r{\mathbf{p}_6}^2},$$

where $\mathbf{p}'_k$s are polynomials in $p,\ q$ of degree k. It is clear that the degree of each iterates can be computed as follows: 0, 1, 2, 5, 8, 13, 18, 25, 32, 41, ...

From the above sequence of the degree, we can arrive at the general formulae: $d_{2n} = 2n^2$ and $d_{2n+1} = 2n^2 + 2n + 1$. Since the degree growth obeys polynomial expression, by using

$$\epsilon(\phi) = \lim_{n\to\infty} \frac{1}{n} log(d_n). \tag{1}$$

one can easily show that the algebraic entropy $\epsilon(\phi)$ is vanishing. This clearly indicates the integrability of the mapping.

Most important application of the algebraic entropy is that this method can be used in combination with singularity confinement to find the non-autonomous forms of the integrable mapping.

For this purpose, one of the basic assumption one can make is that the degree growth of the autonomous and non-autonomous mapping should be the same.

Since the generic mapping is of exponential growth, at some stage of the iteration the degree might be different from that of the autonomous case. At this stage, we need to impose certain restrictions on the parameters to bring down the degree. Solving the constrains of the unknown parameters, we get the non-autonomous integrable mapping.

-As an application, again we consider the same mapping mentioned earlier but now with $a = a(n) = a_n$

$$x_{n+1} + x_{n-1} = \frac{a_n}{x_n} + \frac{1}{x_n^2}.$$

Again by using the projective coordinates, we arrive at each iteration:

$$x_2 = \frac{r^2 + a_1 qr - pq^2}{q^2},$$

$$x_3 = \frac{q\mathbf{Q}_4}{r(r^2 + a_1 qr - pq^2)^2},$$

$$x_4 = \frac{(r^2 + a_1 qr - pq^2)\mathbf{Q}_6}{{\mathbf{Q}_4}^2},$$

$$x_5 = \frac{q\mathbf{Q}_4\mathbf{Q}_{12}}{r(r^2 + a_1 qr - pq^2){\mathbf{Q}_7}^2},$$

where $\mathbf{Q}_k'$s are polynomials in $p,\ q$ of degree k.

Therefore, we obtain the degrees as follows: 0, 1, 2, 5, 10, 21, 42, 85, ... Note that the degree growth is exponential $d_{2m-1} = \dfrac{2^{2m} - 1}{3}$ and $d_{2m} = 2d_{2m-1}$.

Observe that the degree d_5 in the autonomous case is 8 and whereas 10 in the non-autonomous case. As per our assumption, the degree growth should be same for both autonomous and non-autonomous. So, at the iteration x_5 we should impose restrictions on the parameter a_n to bring down the degree from 10 to 8.

The constraints we obtain

$$a_{n+1} - 2a_n + a_{n-1} = 0. \tag{2}$$

Solving (1), we get $a_n = \alpha n + \beta$. Hence, we obtain the non-autonomous mapping

$$x_{n+1} + x_{n-1} = \frac{\alpha n + \beta}{x_n} + \frac{1}{x_n^2}.$$

The degree sequence is exactly same as the autonomous mapping and hence we have vanishing of the algebraic entropy. Thus, we conclude that the non-autonomous mapping as well integrable.

Algebraic entropy for linearizable mapping

Consider the discrete Riccati

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}.$$

By applying the algebraic entropy analysis one can find

$$\{d_n\} = \{1, 1, \dots\}.$$

Obviously, the degree growth is linear and hence the given map is linearizable.

Our second example is

$$x_{n+1} = a x_{n-1} \left(\frac{x_n - a}{x_n - 1} \right),$$

The degrees of this map is $0, 1, 2, 3, 4, \ldots$. Again the degree growth is linear and hence the mapping is linearizable.

Our final example is a non-integrable map

$$x_{n+1} = a x_{n-1} \left(x_n + \frac{1}{x_n} \right).$$

Here, we can find that the degree sequence is $0, 1, 2, 4, 8, 14, 24, 40, 66, 108, \ldots$ Notice that the degree growth is exponential. Therefore, the above map is non-integrable.

In this section, we extend the algebraic entropy method to differential-difference equations. We consider semi-discrete KdV equation:

$$u_{n+1} = u_{n-1} + \frac{u_n'}{u_n}, \tag{3}$$

As before, we start with $u_0 = p$, $u_1 = \frac{q}{r}$, where $deg\ p = 0, deg\ q = 1$, $deg\ r = 1$ and $deg\ t = 1$ and compute the first few iterates of (2). We thus obtain

$$u_2 = \frac{pqr - q'r - qr'}{qr}, \tag{4}$$

$$u_3 = \frac{q^2(pqr - q'r - qr' + r'^2 - rr'') + r^2(p'q^2 + qq'' - q'^2)}{(pqr - q'r - qr')qr}, \tag{5}$$

and so on.

Computing the degree of the successive iterates, we find $d_n = 0,\ 1,\ 2,\ 4,\ 7,\ 11,\ 16,\ 22, \ldots$. i.e. given by $d_n = \dfrac{n^2 - n + 2}{2}$ for $n > 0$. The fact that the degree growth is polynomial and hence the system (2) is integrable.

Singularity confinement

Painlevé test: great heuristic value

How to transpose singularity analysis to the study of discrete systems?

Rational discrete systems have singularities

Do singularities play a role in integrability of discrete systems?

Singularity confinement

characteristic of systems integrable through spectral methods

SC: integrability criterion?

necessary? sufficient?

Difficulties

- Mappings not uniquely defined in both directions

Criterion of pre-image non-proliferation Grammaticos, Ramani and Tamizhmani (1994)

(Eliminates all polynomial nonlinear mappings)

- What do we mean by 'singularity'?

$\rightarrow$"loss of a degree of freedom"

For a mapping $x_{n+1} = f(x_n, x_{n-1})$ this means $\partial x_{n+1}/\partial x_{n-1} = 0$

"Confinement"

mapping recovers the lost degree of freedom

The only way to do this is by the appearance of an indeterminate form $\frac{0}{0}$, $\infty - \infty$, etc., in the subsequent iterations.

Kruskal:

The real problem is the indeterminate form not the simple infinity

Solution

Use continuity with respect to the initial conditions

Introduce a small parameter ϵ

Consider the McMillan mapping:

$$x_{n+1} + x_{n-1} = \frac{2\mu x_n}{1 - x_n^2}$$

Singularity: whenever x passes through ± 1

Assume, x_0 is finite and $x_1 = 1 + \epsilon$

We find:

$x_2 = -\mu/\epsilon - (x_0 + \mu/2) + \mathcal{O}(\epsilon)$,

$x_3 = -1 + \epsilon + \mathcal{O}(\epsilon^2)$

$x_4 = x_0 + \mathcal{O}(\epsilon)$

Singularity confined and mapping recovered memory of the initial conditions through x_0.

Deautonomise the McMillan mapping

$$x_{n+1} + x_{n-1} = \frac{a(n) + b(n)x_n}{1 - x_n^2}$$

Assume: regular x_n and $x_{n+1} = \sigma + \epsilon$ where $\sigma = \pm 1$ We find:

$$x_{n+2} = -\frac{b_{n+1} + \sigma a_{n+1}}{2\epsilon} + \frac{a_{n+1} - \sigma b_{n+1}}{4} - x_n$$

$$x_{n+3} = -\sigma + \frac{2b_{n+2} - b_{n+1} - \sigma a_{n+1}}{b_{n+1} + \sigma a_{n+1}}\epsilon + \mathcal{O}(\epsilon^2) \qquad (1.12)$$

Condition for x_{n+4} to be finite:

$$b_{n+1} - 2b_{n+2} + b_{n+3} + \sigma(a_{n+1} - a_{n+3}) = 0$$

Solution:

$b_n(\equiv z_n) = \alpha n + \beta$ and $a_n = \delta + \gamma(-1)^n$

Ignore even-odd dependence (a=constant)

$$x_{n+1} + x_{n-1} = \frac{a + z_n x_n}{1 - x_n^2}$$

Discrete form of $\mathrm{P_{II}}$!

Similarly:

$$x_{n+1} + x_n + x_{n-1} = a(n) + \frac{b(n)}{x_n}$$

Assume: x_n is regular while x_{n+1} vanishes. We set $x_{n+1} = \epsilon$ and obtain:

$$x_{n+2} = \frac{b_{n+1}}{\epsilon} + a_{n+1} - x_n + \mathcal{O}(\epsilon)$$

$$x_{n+3} = -\frac{b_{n+1}}{\epsilon} + a_{n+2} - a_{n+1} + x_n + \mathcal{O}(\epsilon)$$

x_{n+4} diverges unless $a_{n+3} - a_{n+2} = 0$

For confinement a=constant

$$x_{n+4} = \frac{b_{n+1} - b_{n+2} - b_{n+3}}{b_{n+1}} \epsilon + \mathcal{O}(\epsilon^2)$$

For x_{n+5} finite, second condition:

$$b_{n+1} - b_{n+2} - b_{n+3} + b_{n+4} = 0$$

Solution $b_n = \alpha n + \beta + \gamma(-1)^n$

Ignore even-odd dependence, put $b_n \equiv z_n = \alpha n + \beta$:

$$x_{n+1} + x_n + x_{n-1} = a + \frac{z_n}{x_n}$$

Discrete form of $\mathrm{P_I}$.

Even more interesting result

Deautonomisation of the $f_2 = 0$ QRT

Integrable mapping:

$$x_{n+1}x_{n-1} = \frac{ab(x_n - cq_n)(x_n - dq_n)}{(x_n - a)(x_n - b)}$$

where a, b, c, and d are constants and

$$q_n = q_0\lambda^n$$

The continuous limit is $\mathrm{P}_{\mathrm{III}}$

Not a difference equation,

but a q- (multiplicative) mapping

Today q-discrete forms exist for all d-$\mathbb{P}$'s From geometrical description gave a unified approach to derive discrete Painlevé equations derived certain asymmetric discrete Painlevé equations

Conclusion

In this short review, we briefly pointed out the historical background (certainly not a complete one) of considering the differential equations over complex domain.

The main theme is to the search for differential equations whose general solution is single-valued and analytic. This lead to the discovery of six Painlevé transcendents.

The discovery of singularity confinement and introduction of algebraic entropy have been discussed and demonstrated their effectiveness with examples from maps and differential-difference equations.