On a degenerate algebraic Riccati equation

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Control Problems

Let Z be a Hilbert space and let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on Z. Let U be a Hilbert space called the space of **controls** and let $B : U \to Z$. Let $\zeta \in Z$. Consider the problem:

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The space Z is called the **state space** and $z \in L^2(0, T; Z)$ is the state of the system. We may have several objectives from the point of view of control of the above system.

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- **Exact controllability**
  The system (1) is **exactly controllable** in time $T > 0$, if for any initial data ζ, and any given element ζ_1, there exists a control $u \in L^2(0, T; U)$ such that $z(T) = \zeta_1$.

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All three notions are equivalent if the spaces are finite dimensional.

Optimal Control: Linear Regulator Problem

Let Y be a Hilbert space, called the space of **observation**. Let $C : Z \to Y$ be a bounded linear operator. Let $\zeta \in Z$ be fixed. Let $z(t)$ be the state, *i.e.* the solution of (1). Define the cost functional

$$J(z,u) \;=\; \frac{1}{2}\int_0^T \|Cz(t)\|_Y^2 \, dt + \frac{1}{2}\int_0^T \|u(t)\|_U^2 \, dt.$$

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Problem: Find $u \in L^2(0,T;U)$ such that J is minimized.
This is called an optimal control problem with **finite** time horizon.

Optimal Control: infinite time horizon

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Finite Cost Condition

For every $\zeta \in Z$, there exists a control $u \in L^2(0,\infty;U)$ such that $J(z,u) < \infty$.

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and P satisfies the **algebraic Riccati equation**

$$A^*P + PA - PBB^*P + C^*C \;=\; \mathbf{0}.$$

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$$\min J \;=\; \frac{1}{2}(P\zeta, \zeta)_Z.$$

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Then (1) becomes

$$z'(t) \ = \ (A - BB^*P)z(t), \ t > 0, \ \text{and} \ z(0) \ = \ \zeta.$$

In the case of a finite time horizon, defined by T, we have a **differential** Riccati equation. Given the system (1), we have the dual system:

$$\begin{aligned} -p'(t) &= A^* p(t) + C^* C z, \ 0 < t < T, \\ p(T) &= C^* C z(T). \end{aligned}$$

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$$P(t)\zeta \; = \; p(0).$$

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- If A is exponentially stable, then FCC automatically holds.
- If FCC holds and the pair (A, C) is exponentially detectable, then the solution to the algebraic Riccati equation is unique.

Numerical Approximation

Let A_h, B_h, C_h be finite dimensional approximations to A, B, C respectively using some numerical scheme (eg. Finite element method). Problem: Find $P_h \in \mathcal{L}(\mathbb{R}^N)$ such that $P_h = P_h^* \geq \mathbf{0}$ and

$$A_h^* P_h + P_h A_h - P_h B_h B_h^* P_h + C_h^* C_h \ = \ \mathbf{0}.$$

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Benner: Choose P_0 such that $P_0 = P_0^* \geq \mathbf{0}$ solution to the **degenerate** algebraic Riccati equation:

$$A_h^* P + P A_h - P B_h B_h^* P \;=\; \mathbf{0}$$

which is also such that $A_h - B_h B_h^* P_0$ is exponentially stable.

We are interested in the following problem:
Find $P \in \mathcal{L}(Z)$ such that $P = P^* \geq \mathbf{0}$ such that

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and such that $A - BB^*P$ is exponentially stable.
Let us assume for the time being that the spaces Z and U and Y are **finite dimensional**.

A Comparison Principle

Lemma

Let $C_i \in \mathcal{L}(Z, Y)$ *for* $i = 1, 2$ *be such that* $C_1^* C_1 \geq C_2^* C_2$ *Let* $P_i \in \mathcal{L}(Z)$ *be such that* $P_i = P_i^* \geq \mathbf{0}$ *and*

$$A^* P_i + P_i A - P_i B B^* P_i + C_i^* C_i \;=\; \mathbf{0}$$

for $i = 1, 2$. *If* $A - BB^* P_1$ *is exponentially stable, then* $P_1 \geq P_2$ ■

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for $i = 1, 2$. If $A - BB^* P_1$ is exponentially stable, then $P_1 \geq P_2$ ■

Corollary

The algebraic and degenerate algebraic Riccati equations admit at most one solution P such that $A - BB^* P$ is exponentially stable. In particular, if A is itself exponentially stable, then the degenerate equation has no non-trivial solutions such that $A - BB^* P$ is exponentially stable. ■

A Special Case

Theorem

The following are equivalent:
(i) The operator $-A$ is exponentially stable and there exists $\alpha > 0$ such that

$$\int_0^\infty \|B^* e^{-tA^*} z\|_Z^2 \, dt \;\geq\; \alpha \|z\|_Z^2 \tag{2}$$

for all $z \in Z$.

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(ii) The degenerate algebraic Riccati equation admits solution $P \in \mathcal{L}(Z)$ which is **invertible** *and such that $A - BB^*P$ is exponentially stable.*

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Proof: Step 1: If $-A$ is exponentially stable and (2) holds,

$$Q \;=\; \int_0^\infty e^{-tA} BB^* e^{-tA^*} \, dt$$

is well defined and $Q = Q^*$. Further, for any $z \in Z$, (2) implies that

$$(Qz, z)_Z \;\geq\; \alpha \|z\|_Z^2.$$

Thus, Q is invertible and $Q \geq \mathbf{0}$.

Step 2. Let $Q(t) = e^{-tA}BB^*e^{-tA^*}$. Then

$$BB^* = Q(0) = -\int_0^\infty \frac{d}{dt}Q(t)\,dt$$

and so we deduce that

$$AQ + QA^* = BB^*.$$

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Step 3. Since P is invertible, and solves the degenerate equation, we see that

$$P(A - BB^*P)P^{-1} \ = \ -A^*$$

and the RHS is, by assumption, also exponentially stable. Thus $A - BB^*P$ is also exponentially stable.
Step 4. The converse is proved by essentially retracing this proof. ■

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Remark 2. If the pair $(-A, B)$ is exactly controllable in some time $T > 0$, then there exists $\alpha > 0$ such that

$$\int_0^T \|B^* e^{-tA^*} z\|_Z^2 \, dt \ \geq \ \alpha \|z\|_Z^2$$

for all $z \in Z$ and so (2) is also true. Thus, the above theorem is applicable if $-A$ is exponentially stable and the pair $(-A, B)$ is exactly controllable.

A Variational Characterization

We will assume that $-A$ is exponentially stable and that (2) holds. Let P be the solution to the degenerate algebraic Riccati equation obtained in the proof of Theorem 1.

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Lemma

Let H be a real Hilbert space. Let $\{G_n\}$ be a sequence in $\mathcal{L}(H)$ such that $G_n = G_n^* \geq \mathbf{0}$. Assume, further that, for every $v \in H$, the sequence $\{(G_n v, v)_H\}$ is decreasing. Then, there exists $G \in \mathcal{L}(H)$ such that $G = G^* \geq \mathbf{0}$ and, for every $v \in H$, $G_n v \to Gv$ in H. ■

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Since $A - BB^*P$ is exponentially stable, (A, I) is exponentially detectable. So is the pair (A, kI) for any $k \in \mathbb{R}$. In particular, for every $\varepsilon > 0$, there exists a unique $P_\varepsilon = P_\varepsilon^* \geq \mathbf{0}$ such that

$$P_\varepsilon A + A^* P_\varepsilon - P_\varepsilon BB^* P_\varepsilon + \varepsilon^2 I \;=\; \mathbf{0}.$$

Further, $A - BB^*P_\varepsilon$ is exponentially stable.
Then, by lemma, there exists $P_0 = P_0^* \geq \mathbf{0}$ solution of the degenerate algebraic Riccati equation.

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Then, by lemma, there exists $P_0 = P_0^* \geq \mathbf{0}$ solution of the degenerate algebraic Riccati equation.
Since $A - BB^*P$ is exponentially stable, we get, by the comparison principle, that $P \geq P_0$. Again, by the same principle, we have $P_\varepsilon \geq P$ and, passing to the limit, $P_0 \geq P$. Thus, $P_0 = P$.

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Let $\zeta \in Z$ be fixed such that $\zeta \neq 0$.
Now, for $u \in L^2(0, \infty; U)$, set z_u to be the solution of (1). Define

$$E_\zeta \;=\; \{u \in L^2(0,\infty;U) \mid z_u \in L^2(0,\infty;Z\}.$$

Consider

$$\min_{u \in E_\zeta} \int_0^\infty \|u(t)\|_U^2 \, dt.$$

FCC $\Rightarrow E_\zeta \neq \emptyset$.
E_ζ closed?

Proposition

If $-A$ is exponentially stable and (2) holds, then the above minimization problem admits a solution. We have

$$(P\zeta,\zeta)_Z \;=\; \min_{u\in E_\zeta}\int_0^\infty \|u(t)\|_U^2\,dt$$

and the minimizer is given by

$$u(t) \;=\; -B^* e^{-tA^*}P\zeta. \;\blacksquare$$

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Remark 3. Since $-A$ is exponentially stable, A is NOT and so $0 \notin E_\zeta$.

The General Case

Henceforth, we will assume the following to hold:
(H) There exist subspaces Z_s and Z_u of Z such that
(i) $Z = Z_s \oplus Z_u$.
(ii) Z_s and Z_u are invariant under A.
(iii) The restriction of A to Z_s is exponentially stable.
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Example The matrix A has no eigenvalues on the imaginary axis. Then we can find invariant subspaces Z_s and Z_u such that all the eigenvalues of the restriction of A to Z_s are with negative real part and all the eigenvalues of the restriction of A to Z_u have positive real part. ■

Let $\pi_s : Z \to Z_s$ and $\pi_u : Z \to Z_u$ be the canonical projections with respect to this decomposition of Z.

$$\pi_u A \;=\; A\pi_u \;=\; \pi_u A \pi_u.$$

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Assume that the pair $(\pi_u A, \pi_u B)$ is such that there exists $\alpha > 0$ satisfying:

$$\int_0^\infty \|(\pi_u B)^* e^{-t(\pi_u A)^*} z\|_Z^2 \, dt \geq \alpha \|z\|_Z^2 \tag{3}$$

for all $z \in Z_u$.

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for all $z \in Z_u$. Then, by Theorem 1, there exists $P_u \in \mathcal{L}(Z_u)$ such that $P_u = P_u^* \geq \mathbf{0}$ and

$$P_u(\pi_u A) + (\pi_u)^* P_u - P_u(\pi_u B)(\pi_u B)^* P_u \;=\; \mathbf{0}.$$

Further, $\pi_u A - (\pi_u B)(\pi_u B)^* P_u$ is exponentially stable.

Theorem

Assume that the hypothesis (H) holds and that (3) is true. Let $P_u \in \mathcal{L}(Z_u)$ be as detailed earlier. Define

$$P = \pi_u^* P_u \pi_u.$$

Then $P = P^* \geq \mathbf{0}$; P satisfies the degenerate algebraic Riccati equation and $A - BB^*P$ is exponentially stable.

Theorem

Assume that the hypothesis (H) holds and that (3) is true. Let $P_u \in \mathcal{L}(Z_u)$ be as detailed earlier. Define

$$P \;=\; \pi_u^* P_u \pi_u.$$

Then $P = P^* \geq \mathbf{0}$; P satisfies the degenerate algebraic Riccati equation and $A - BB^*P$ is exponentially stable.

Proof: Clearly P is self-adjoint and non-negative. That it satisfies the degenerate algebraic Riccati equation follows by multiplying the equation for P_u on the left by π_u^* and on the right by π_u and using the fact that π_u commutes with A (and so its adjoint commutes with A^*) and that π_u is a projection.

Finally we see that (with respect to the decomposition $Z = Z_s \oplus Z_u$),

$$\begin{bmatrix} \pi_s(A - BB^*P)z \\ \pi_u(A - BB^*P)z \end{bmatrix} =$$

$$= \begin{bmatrix} \pi_s A & -\pi_s BB^*\pi_u^* P_u \\ \mathbf{0} & \pi_u A - \pi_u BB^*\pi_u^* P_u \end{bmatrix} \begin{bmatrix} \pi_s z \\ \pi_u z \end{bmatrix}.$$

Since both diagonal blocks of the upper triangular matrix are exponentially stable, it follows that $A - BB^*P$ is also exponentially stable. ∎

Assume that A has no eigenvalues on the imaginary axis.
If $(-A, B)$ is exactly controllable in time $T > 0$, then so is $(-\pi_u A, \pi_u B)$.
The eigenvalues of $\pi_s A$ are precisely those of A with negative real part.
Since $\pi_u A - \pi_u BB^* \pi_u^* P_u$ is similar to $-(\pi_u A)^*$, the eigenvalues of this matrix are the reflections on the imaginary axis of those of A with positive real part.
Thus, the eigenvalues of $A - BB^* P$ are those of A with negative real part and the reflections on the imaginary axis of those eigenvalues of A with positive real part.

One dimensional case

Degenerate Equation:

$$2ap - b^2p^2 = 0.$$

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When $a < 0$, $p_\varepsilon \to p = 0$.
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When $a < 0$, $p_\varepsilon \to p = 0$.
When $a > 0$, $p_\varepsilon \to p = \frac{2a}{b^2}$.
Thus,

$$a - b^2p \;=\; \begin{cases} a & \text{when } a < 0 \\ -a & \text{when } a > 0. \end{cases}$$

Infinite Dimensions

Let Z be a Hilbert space and let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup which is exponentially stable. Thus, there exists $c > 0$ such that, for all $t \geq 0$, we have

$$\|e^{tA}\| \leq Me^{-ct}.$$

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Let $\alpha > 0$. We define

$$(-A)^{-\alpha}z = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{tA}z\,dt,\ z \in Z.$$

Then $(-A)^{-\alpha} \in \mathcal{L}(Z)$.

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Then $(-A)^{-\alpha} \in \mathcal{L}(Z)$.
If $0 \leq \alpha \leq 1$, we set

$$(-A)^\alpha = (-A)(-A)^{\alpha-1}$$

with the domain

$$D((-A)^\alpha) = \{z \in Z \mid (-A)^{\alpha-1} z \in D(A)\}.$$

We have

$$e^{tA}Z \subset D((-A)^\alpha),\ (-A)^\alpha e^{tA} \in \mathcal{L}(Z) \text{ for all } t > 0$$

and there exists $k > 0$ and $C(\alpha) > 0$ such that

$$\|(-A)^\alpha e^{tA}\| \ \leq \ C(\alpha)t^{-\alpha}e^{-kt},\ t \geq 0.$$

Unbounded Control Operator

Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup on a Hilbert space Z. Let A^* denote the Z-adjoint of A and let $(D(A^*))'$ denote the dual of $D(A^*)$ with respect to the Z-topology. Let $B \in \mathcal{L}(U, (D(A^*)')$.

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Let $\lambda \in \rho(A)$, the resolvent set of A. Then $(\lambda I - A) \in \mathcal{L}(D(A), Z)$ and has a bounded inverse in Z. Further $(\lambda I - (A^*)^*)$, the extension of $(\lambda I - A)$ to $D(A^*))'$, also denoted by $(\lambda I - A)$ has a bounded inverse from $(D(A^*))'$ into Z. Thus, there exists $B_0 \in \mathcal{L}(U, Z)$ such that $B = (\lambda I - A)B_0$.

The Parabolic Case

Let A be an analytic semigroup and let $A - \lambda_0 I$ be exponentially stable.Let $B_1 \in \mathcal{L}(U, Z)$ and $0 < \alpha < 1$ be such that

$$B = (\lambda_0 I - A)^{1-\alpha} B_1$$

in the sense explained earlier.
In other words, we have $B \in \mathcal{L}(U, (D(A^*)')$ and

$$B_1 = (\lambda_0 I - A)^{\alpha - 1} B \in \mathcal{L}(U, Z).$$

Degenerate Riccati Equation

Find $P \in \mathcal{L}(Z)$ such that $P = P^* \geq \mathbf{0}$ and

$$A^*P + PA - P(\lambda_0 I - A)^{1-\alpha} B_1 B_1^* (\lambda_0 I - A^*)^{1-\alpha} P \ = \ \mathbf{0}$$

and such that

$$A - (\lambda_0 I - A)^{1-\alpha} B_1 B_1^* (\lambda_0 I - A^*)^{1-\alpha} P$$

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and such that

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is exponentially stable.

The equation is interpreted as follows: for all $\xi, \eta \in D(A)$,

$$(P\xi, A\eta) + (PA\xi, \eta) - (PBB^*P\xi, \eta) \;=\; 0.$$

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- $Z_u \cap D(A) = Z_u$ and $Z_s \cap D(A)$ invariant under A.
- $A|_{Z_s}$ and $-A|_{Z_u}$ exponentially stable.
- Let π_u and π_s be the projections of Z onto Z_u and Z_s respectively. There exists $\beta > 0$ such that

$$\int_0^\infty \|(\pi_u B)^* e^{-t(\pi_u A)^*} z\|_U \, dt \ \geq \ \beta \|z\|_Z^2$$

for all $z \in Z_u$.

Theorem

Under the preceding hypotheses, there exists $P = P^* \geq \mathbf{0}$ *in* $\mathcal{L}(Z)$ *such that*
(i)

$$A^*P + PA - PBB^*P = \mathbf{0},$$

(ii) $A - BB^*P$ *is exponentially stable and*
(iii) $P \in \mathcal{L}(D(A), D(A^*))$.

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(iii) $P \in \mathcal{L}(D(A), D(A^*))$.

Proof: Z_u is finite dimensional and we have $P_u \in \mathcal{L}(Z_u)$ as in Theorem 2. We can set $P = \pi_u^* P_u \pi_u$ and verify that (i) and (ii) are true. We will prove (iii).

$$
\begin{aligned}
\pi_u B &= \pi_u(\lambda_0 I - A)(\lambda_0 I - A)^{-\alpha} B_1 \\
&= (\lambda_0 I - A)\pi_u(\lambda_0 I - A)^{-\alpha} B_1.
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But $(\lambda_0 I - A)^{-\alpha} B_1 \in \mathcal{L}(U, Z)$ and so $\pi_u(\lambda_0 I - A)^{-\alpha} B_1 \in \mathcal{L}(U, Z_u)$. Since Z_u is finite dimensional $A|_{Z_u} \in \mathcal{L}(Z_u)$ and so $\pi_u B \in \mathcal{L}(U, Z_u)$.

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$$
PBB^*P = (\pi_u^* P)(\pi_u B)(\pi_u B)^*(P\pi_u) \in \mathcal{L}(Z).
$$

Let $x \in D(A)$. Since P solves the equation, we easily deduce that $Px \in D(A^*)$.

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$$(A^* Px, y) \;=\; (PBB^* Px, y) - (PAx, y).$$

$$(PAx, y) \;=\; (\pi_u^* P_u \pi_u Ax, y) = (\pi_u^* P_u A \pi_u x, y)$$

whence since $A|_{Z_u} \in \mathcal{L}(Z_u)$,

$$|(PAx, y)| \;\leq\; C\|x\|_Z\|y\|_Z.$$

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$$|((PBB^* - \lambda_0 I)Px, y)| \;\leq\; C\|x\|_Z\|y\|_Z.$$

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Thus

$$|((A^* - \lambda_0 I)Px, y)| \leq C\|x\|_Z \|y\|_Z$$

which gives

$$\|Px\|_{D(A^*)} \leq C\|x\|_Z \leq C\|x\|_{D(A)}$$

for all $x \in D(A)$. Thus $P \in \mathcal{L}(D(A), D(A^*))$ $\blacksquare$.

Variational Characterization

Let $\zeta \in Z$ and let $z_{\zeta,v}$ be the solution of

$$\begin{aligned} z'(t) &= Az(t) + Bv(t),\ \ t > 0, \\ z(0) &= \zeta. \end{aligned}$$

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Let

$$E_\zeta = \left\{ v \in L^2(0,\infty;U) \mid \ z_{\zeta,v}(t) \overset{z_{\zeta,v}\in L^2(0,\infty;Z)}{\to} 0 \text{ as } t \to \infty \right\}.$$

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Theorem

Let $\zeta \in D(A)$. Then $E_\zeta \neq \emptyset$ and

$$(P\zeta,\zeta) \ = \ \min_{v \in E_\zeta} \int_0^\infty \|v(t)\|_U^2 \, dt$$

and the optimal solution is given by

$$v(t) \ = \ -B^*Pe^{t(A-BB^*P)}\zeta.$$

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and $z^u_{\zeta,v}$ is the solution of

$$\begin{aligned} z'(t) &= \pi_u A z(t) + \pi_u B v(t), \;\; t > 0, \\ z(0) &= \pi_u \zeta. \end{aligned}$$

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and $z_{\zeta,v}^u$ is the solution of

$$\begin{aligned} z'(t) &= \pi_u Az(t) + \pi_u Bv(t),\ t>0, \\ z(0) &= \pi_u\zeta. \end{aligned}$$

Proof:

$$(P\zeta,\zeta) \;=\; (\pi_u^* P_u \pi_u \zeta, \zeta) \;=\; (P_u\pi_u\zeta, \pi_u\zeta)$$

and the result follows from the finite-dimensional version. ■

Approximation

Since $A - BB^*P$ is exponentially stable, the pair (A, I) is exponentially detectable. Thus for every $\varepsilon > 0$, we have a unique $P_\varepsilon \in \mathcal{L}(Z)$ with $P_\varepsilon = P_\varepsilon^* \geq \mathbf{0}$ and

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Further $A - BB^* P_\varepsilon$ is exponentially stable.
By the comparison principle $(P_\varepsilon \zeta, \zeta)$ decreases as $\varepsilon \downarrow 0$ for all $\zeta \in Z$ and so there exists $P_0 \in \mathcal{L}(Z)$ such that $P_0 = P_0^* \geq \mathbf{0}$ and $P_\varepsilon \zeta \to P_0 \zeta$ for all $\zeta \in Z$.

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By the comparison principle $(P_\varepsilon \zeta, \zeta)$ decreases as $\varepsilon \downarrow 0$ for all $\zeta \in Z$ and so there exists $P_0 \in \mathcal{L}(Z)$ such that $P_0 = P_0^* \geq \mathbf{0}$ and $P_\varepsilon \zeta \to P_0 \zeta$ for all $\zeta \in Z$.
Question: $P_0 = P(= \pi_u^* P_u \pi_u)$?
By the comparison principle we get $P_\varepsilon \geq P$ and so, for $\zeta \in Z$, we have

$$(P_0 \zeta, \zeta) \ \geq \ (P\zeta, \zeta).$$

Let $\zeta \in D(A)$. Then

$$(P_\varepsilon \zeta, \zeta) \;=\; \min_{v \in E_\zeta} \left\{ \varepsilon^2 \int_0^\infty \|z(t)\|_Z^2 \; dt + \int_0^\infty \|v(t)\|_U^2 \; dt \right\}.$$

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Thus, for any fixed $v \in E_\zeta$, we get, on passing to the limit,

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Thus, for all $\zeta \in D(A)$, we have $(P_0\zeta, \zeta) = (P\zeta, \zeta)$ and, by density, it also holds for all $z \in Z$, which proves that $P_0 = P$.

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Remark; When B is bounded, we can independently prove that $P = P_0$ as and the variational characterization. When B is unbounded we need stronger convergence properties of $P_\varepsilon\zeta$ to pass to the limit in the term $P_\varepsilon BB^*P_\varepsilon$. So we first prove the variational characterization and use it to show that $P_0 = P$.

Finally, when B is bounded, we can prove the infinite dimensional analogue of of the theorem where $-A$ is exponentially stable.

Theorem

*Let $B \in \mathcal{L}(U, Z)$. Let A be the infinitesimal generator of a $\mathcal{C}_0$-**group** and assume that $-A$ is exponentially stable. Assume that there exists $\beta > 0$ such that for all $z \in Z$,*

$$\int_0^\infty \left\| B^* e^{-tA^*} z \right\|_U^2 \, dt \;\geq\; \beta \|z\|_Z^2.$$

Then, there exists $P \in \mathcal{L}(Z), P = P^* \geq \mathbf{0}$ which is invertible and such that
(i) P maps $D(A)$ onto $D(A^*)$ (and so P^{-1} maps $D(A^*)$ onto $D(A)$).
(ii) $PA + A^*P - PBB^*P = \mathbf{0}$.
(iii)$A - BB^*P$ is exponentially stable.

Proof:

Step 1 Define

$$Q = \int_0^\infty e^{-tA} B B^* e^{-tA^*} \; dt.$$

then Q is well-defined, $Q = Q^* \geq \mathbf{0}$ and by hypothesis

$$(Qz, z) \;\geq\; \beta \|z\|_Z^2.$$

By Lax-Milgram, Q is invertible. Set $P = Q^{-1}$.

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By Lax-Milgram, Q is invertible. Set $P = Q^{-1}$.
Step 2 Let $y, z \in D(A^*)$. Then

$$(Qy, A^*z) + (A^*y, Qz)\ =\ (B^*y, B^*z).$$

In particular,

$$|(Qy, A^*z)|\ \leq\ C\|z\|_Z.$$

Thus $Qy \in D(A)$ and we can formally write

$$AQ + QA^*\ =\ BB^*.$$

Step 3 We have $PAQ + A^* = PBB^*$. Thus

$$-A^* \;=\; P(A - BB^*P)P^{-1}$$

and so $A - BB^*P$ is exponentially stable. If $y \in D(A^*)$, then $Qy \in D(A)$ and

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$$PA(Qy) + A^*y - PBB^*P(Qy) = \mathbf{0}.$$

Step 4 Further (A, I) is exponentially detectable and so for every $\varepsilon > 0$, we have $P_\varepsilon \in \mathcal{L}(Z), P_\varepsilon = P_\varepsilon^* \geq \mathbf{0}, A - BB^*P_\varepsilon$ exponentially stable and

$$P_\varepsilon A + A^*P_\varepsilon - P_\varepsilon BB^*P_\varepsilon + \varepsilon^2 I = \mathbf{0}.$$

Step 3 We have $PAQ + A^* = PBB^*$. Thus

$$-A^* \;=\; P(A - BB^*P)P^{-1}$$

and so $A - BB^*P$ is exponentially stable. If $y \in D(A^*)$, then $Qy \in D(A)$ and

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That is, if $x \in D(A)$, then $P_\varepsilon x \in D(A^*)$ and

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In particular, for $y \in D(A^*)$, we have $Qy \in D(A)$ and so $P_\varepsilon(Qy) \in D(A^*)$ and

$$P_\varepsilon A(Qy) + A^*P_\varepsilon(Qy) - P_\varepsilon BB^*P_\varepsilon(Qy) + \varepsilon^2 Qy \;=\; 0.$$

From the above two equations for Qy and the comparison principle, we have

$$((P_\varepsilon - P)Qy, Qy) \ \geq \ 0$$

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Step 5 As usual there exists $P_0 \in \mathcal{L}(Z), P_0 = P_0^* \geq \mathbf{0}$ such that $P_\varepsilon z \to P_0 z$ for every $z \in Z$. Then (since B is bounded)

$$P_0 A + A^* P_0 - P_0 B B^* P_0 \;=\; \mathbf{0}.$$

That is if $x \in D(A)$, we have $P_0 x \in D(A^*)$ and

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Further,

$$((P_0 - P)Qy, Qy) \ \geq 0, \ y \in D(A^*).$$

Again from the comparison principle, since $A - BB^*P$ is exponentially stable, we get

$$((P - P_0)Qy, Qy) \ \geq \ 0, \ y \in D(A^*).$$

Thus, $((P - P_0)Qy, Qy) = 0$ for all $y \in D(A^*)$ and so for all $y \in Z$ by density. Since Q is invertible, we have $((P - P_0)z, z) = 0$ for all $z \in Z$ and so $P = P_0$. Thus P solves the degenerate Riccati equation, which shows that $P : D(A) \to D(A^*)$ and we already saw that $Q : D(A^*) \to D(A)$ and so these maps are onto. ■

Thank You!