The dynamics of holomorphic correspondences on compact Riemann surfaces

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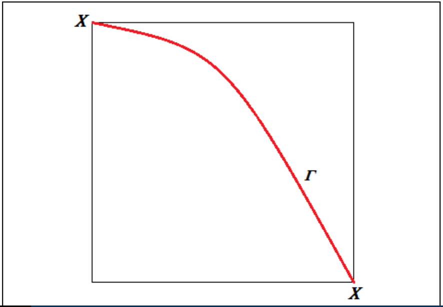

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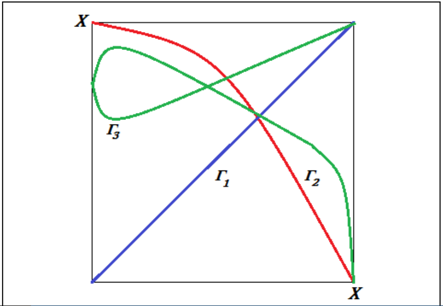

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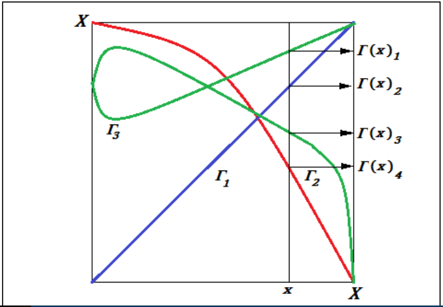

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But not too hard to show that there are infinitely many holomorphic correspondences Γ on a compact hyperbolic Riemann surface; even satisfying

$$d_{top}(\Gamma) \ > \ d_{top}({}^{\dagger}\Gamma).$$

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- Theorem B (time permitting) will address the issue in the box above.

Composing two holomorphic correspondences

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To code the k-chain data into the above “composition” we need to do some work. . .

Composing two holomorphic correspondences, cont'd.

To begin with, we now use an alternative representation:

$$\mathit{\Gamma}^1 \;=\; {\sum}'_{1\leq j\leq L_1} \Gamma^{\bullet}_{1,\,j}, \quad \mathit{\Gamma}^2 \;=\; {\sum}'_{1\leq j\leq L_2} \Gamma^{\bullet}_{2,\,j},$$

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We then define

$$\mathit{\Gamma}^2 \circ \mathit{\Gamma}^1 \;:=\; \sum_{j=1}^{L_1}\sum_{l=1}^{L_2}\sum_{S\in\mathcal{S}(j,\,l)}\nu_S S,$$

where :

$\mathcal{S}(j,l)$:= set of distinct irred. components of $\Gamma^{\bullet}_{2,\,l}\star\Gamma^{\bullet}_{1,\,j}$.

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$\nu_S :=$ **generic** no. of y's – as (x, z) varies through S – for which the memberships given in $(*)$ hold.

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A point z_0 belongs to the Fatou set if there exists a nbhd. $U \ni z_0$ such that for *every* infinite path $(z_0, z_1, z_2, \dots; \alpha_1, \alpha_2, \alpha_3, \dots)$, each sequence of analytic germs of $\Gamma^{\bullet}_{(\alpha_1, \dots, \alpha_n)}$ at $(z_0, z_1, \dots, z_n)$, $n = 1, 2, 3, \dots$, determined by lifting U into these varieties admits

The Fatou set: preliminary notations/notions

Fix a compact Riemann surface X and a correspondence Γ on it.

- $\boldsymbol{N}$**-path:**
  $(z_0, \dots, z_N;\, \alpha_1, \dots, \alpha_N) \in X^{N+1} \times [1\,..\,L]^N \,:\, (z_{j-1}, z_j) \in \Gamma^{\bullet}_{\alpha_j},\ j \leq N.$
- $\mathscr{P}_{\boldsymbol{N}}(\boldsymbol{z_0}) =$ set of all N-paths starting at z_0.
- Given a multi-index $\boldsymbol{\alpha} \in [1\,..\,L]^j$
  $\Gamma^{\bullet}_{\boldsymbol{\alpha}} := \{(x_0, \dots, x_j) \in X^{j+1} : (x_{i-1}, x_i) \in \Gamma^{\bullet}_{\alpha_i},\ 1 \leq i \leq j\}.$

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The Fatou set: secondary notations

A useful map:

$$\pi_j^{(k)} : X^{k+1} \longrightarrow X, \qquad \pi_j^{(k)} : (z_0, z_1, \ldots, z_k) \longmapsto z_j,\ 0 \leq j \leq k.$$

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$$\begin{aligned} \boldsymbol{\mathcal{Z}}_{[j]} &:= (z_0, \dots, z_j; \alpha_1, \dots, \alpha_j), && 1 \leq j \leq N \\ \mathrm{pre}(\boldsymbol{\mathcal{Z}}) &:= \boldsymbol{\mathcal{Z}}_{[N-1]}, && N \geq 2. \end{aligned}$$

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Next, we define **sequences of analytic germs:**

$\mathscr{S}(U, \boldsymbol{\mathcal{Z}}) :=$ set of irred. components of

$\Gamma^{\bullet}_{\boldsymbol{\alpha}} \cap (U \times X)$ containing Z, $\qquad$ if $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_1(z_0)$,

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S is an element of $\mathscr{S}(U, \mathsf{pre}(\boldsymbol{\mathcal{Z}}))$,

Fundamental compositional relation

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Definition (The Fatou set)

A point z_0 is said to belong to the *Fatou set of Γ* if there exists a **single** connected open nbhd. $U \ni z_0$ such that for each $n \in \mathbb{Z}_+$, each $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$ admits an analytic branch $(\mathscr{A}_1, \dots, \mathscr{A}_n; U)$ of Γ along $\boldsymbol{\mathcal{Z}}$, and

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$$\mathscr{F}(z_0) := \Big\{ \pi_0^{(n)} \times \pi_n^{(n)}(\mathscr{A}_n) : n \in \mathbb{Z}_+,\ \boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0),\ \text{and } (\mathscr{A}_1, \dots, \mathscr{A}_n; U) \\ \text{is an analytic branch of } \Gamma \text{ along } \boldsymbol{\mathcal{Z}} \Big\},$$

viewed as a set comprising currents of integration, is relatively compact in the space of $(1,1)$-currents on $U \times X$.

The iterative tree

Suppose, for $z_0 \in X$, $\exists U \ni z_0$, a connected nbhd. of z_0, such that for each $n \in \mathbb{Z}_+$, each $\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)$ admits an analytic branch $(\mathscr{A}_1, \dots, \mathscr{A}_N; U)$ of Γ along $\boldsymbol{\mathcal{Z}}$. We can define an infinite tree $\tau(\Gamma, U)$ as follows.

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$$V(\tau(\Gamma, U)) := \bigcup_{n \in \mathbb{Z}_+} \bigcup_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \mathscr{S}(U, \boldsymbol{\mathcal{Z}}),$$

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\end{aligned}$$

Such a tree is called *the iterative tree at* z_0.

The Fatou set: **very** basic properties

Unlike the case with rational maps, $\mathscr{F}(\Gamma)$ and $\text{supp}(\mu_\Gamma)$ **do not,** in general, partition X under the condition

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This is the motivation of Theorem B, which we shall see soon.

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So, for instance, viewing a smooth (k,k)-form Ω as a current, and a test function as a $(0,0)$-form,

$$\langle \mathit{\Gamma}^*(\Omega), \varphi \rangle := \sum_{j=1}^{N} m_j \int_{\mathrm{reg}(\Gamma_j)} \left(\pi_1|_{\Gamma_j} \right)^* \varphi \left(\pi_2|_{\Gamma_j} \right)^* \Omega.$$

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- comes from dualising $(\pi_1)_*$,
- is the interpretation of "$(\pi_2^*(\Omega) \wedge [\mathit{\Gamma}])$" in this case.

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Let ω_X denote the normalized Kähler form associated to the hyperbolic metric. Call $d_{top}(\Gamma) =: d_1$ and $d_{top}({}^\dagger\Gamma) =: d_0$. Easy to show that

$$\langle (\Gamma^n)^*(\omega_X), \varphi \rangle = \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \sum_{\mathscr{A} \in \mathscr{S}(U, \boldsymbol{\mathcal{Z}})} \int_{\mathrm{reg}(\widetilde{\mathscr{A}})} \left(\pi_1|_{\Gamma_j} \right)^* \varphi \left(\pi_2|_{\Gamma_j} \right)^* \omega_X,$$

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where $\widetilde{\mathscr{A}} = \pi_0^{(n)} \times \pi_n^{(n)}(\mathscr{A})$. Thus:

$$d_1^{-n}\left|\langle (\Gamma^n)^*(\omega_X), \varphi\rangle\right| \leq d_1^{-n} \sup|\varphi| \sum_{\boldsymbol{\mathcal{Z}} \in \mathscr{P}_n(z_0)} \sum_{\mathscr{A} \in \mathscr{S}(U, \boldsymbol{\mathcal{Z}})} \int_{\mathrm{reg}(\widetilde{\mathscr{A}})} \left(\pi_2|_{\Gamma_j}\right)^* \omega_X,$$

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Let (X_1,ω_1) and (X_2,ω_2) be compact k-dim'l. Kähler manifolds, and let U be a relatively compact open subset of X_1. Let $\boldsymbol{F}$ be a family of reduced, irreducible, analytic subsets of $U\times X_2$ of pure dimension $p : 1\leq p\leq k$.

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- (a) The volumes of the sets in $\boldsymbol{F}$ are uniformly bounded; and
- (b) Given a compact $K\subset U$, there $\exists C_K>0$ such that, for $\mathscr{A},\mathscr{B}\in\boldsymbol{F}$, $\mathscr{A}\cap(K\times X_2)$ and $\mathscr{B}\cap(K\times X_2)$ are no farther than C_K in the Hausdorff metric.

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Thus, from $(**)$, we have that

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$$\int_X \varphi\,d\mu_\Gamma \;=\; \lim_{n\to\infty} d_1^{-n}\langle(\Gamma^n)^*(\omega_X),\varphi\rangle \;=\; 0.$$

Hence the result. ∎

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- The n-th generation of $\widetilde{\tau}(\Gamma, U)$ contains at least $(1 - \varepsilon^n)\, d_0^n$ vertices from the n-th generation of the iterative tree; and

Theorem B: preliminaries

Observe: $(**)$ suggests that one could allow the volumes of branches to grow at a certain exponential rate. This motivates the following:

We say that *most analytic branches of Γ around z_0 converge* if there exist a connected nbhd. $U \ni z_0$ that admits an iterative tree $\tau(\Gamma, U)$, and an $\varepsilon \in (0,1)$ such that there is a connected subtree $\widetilde{\tau}(\Gamma, U)$, and so that

- The n-th generation of $\widetilde{\tau}(\Gamma, U)$ contains at least $(1-\varepsilon^n)\, d_0^n$ vertices from the n-th generation of the iterative tree; and
- The family

$$\overline{\mathscr{F}}(z_0) := \{\pi_0^{(n)} \times \pi_n^{(n)}(\mathscr{A}_n) : n \in \mathbb{Z}_+, \text{ and } \mathscr{A}_n \in V_n(\widetilde{\tau}(\Gamma, U))\}$$

is relatively compact in the space of $(1,1)$-currents on $U \times X$.

Theorem B

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Let X be a compact Riemann surface. Let Γ and μ_Γ be as in Theorem A. Suppose the postcritical set of Γ is disjoint from $\operatorname{supp}(\mu_\Gamma)$.

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The proof of

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follows from the fact that there is at most exponential volume-growth of analytic branches of Γ, and that $\overline{\mathscr{F}}(\Gamma)$ does not depend on the size of ε a long as $0 < \varepsilon < 1$. We then apply $(**)$.