Derived birational Invariants and Unramified Cohomology

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(jt. w/ B. KAHN)

• Study of birational invariants an old and venerated topic in algebraic geometry.

• Eg: Unramified Cohomology
X a smooth, projective algebraic variety over a field k, [redacted] an integer invertible in F. For every (Zariski) open $U \subseteq X$,

$$U \longmapsto H^i_{et}(U, \mu_n^{\otimes r})$$

defines a Zariski presheaf, the associated sheaf is denoted $\mathcal{H}^i(\mu_n^{\otimes j})$.

Global sections $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ is a birational invariant.

- Unramified cohomology with coefficients in $\mu_n^{\otimes i}$; shorter notation $H^i_{nr}(X, \mu_n^{\otimes i})$.
- $H^0(X, \mathcal{H}^i(\mu_n^{\otimes i})) \subseteq H^i(k(X), \mu_n^{\otimes i})$; consists of elements which at any point $P \in X$ of codimension 1 and local ring $\mathcal{O}_{X,P}$, residue field κ_P, comes from $H^i(\mathcal{O}_{X,P}, \mu_n^{\otimes i})$ under the natural map

$$H^i(\mathcal{O}_{X,P}, \mu_n^{\otimes i}) \longrightarrow H^i(k(X), \mu_n^{\otimes i}).$$

- Can also be described in terms of valuations on $k(X)$ and the associated 'residue' maps.

Eg: $H^0(X, \mathcal{H}^1(\mu_n)) : {}_n\mathrm{Pic}(X)$ } n-torsion in Picard gp.
$H^0(X, \mathcal{H}^2(\mu_n)) : {}_n\mathrm{Br}(X)$ } n-torsion in Brauer gp.

- $X/_k$ s.t. $k(X)$ is purely transcendental over k.

Then $H^i_{nr}(X, \mu_n^{\otimes i}) \cong H^i(\mathrm{Gal}(k_s/_k), \mu_n^{\otimes i})$

Thus these are "stable birational invariants."

→ $X/_k$ stably birational over k to projective space, then "know" these unramified cohomology gps.

→ Next Simplest: Anisotropic quadrics $/_k$. Quadrics $X/_k$, smooth projective s.t. $X(k) = \phi$. $H^i_{nr}(X, \mu_n^{\otimes i})$?
(cf. KAHN-S)

- See also Colliot-Thélène–Ojanguren, Peyre, Kahn. (Projective homogeneous varieties ...).

AIM OF TODAY'S LECTURE:

- These unramified cohomology groups are just "the tip of an iceberg", and that there are in fact a whole array of related birational invariants, when viewed from the standpoint of Categories and Voevodsky's motives.
- We will try to give a sample of this array and how they fit with other classical invariants.
- Freely acknowledge that our results are computably sparse, but we find it fascinating that there exists a whole array of these birational invariants, though their exact computation is a challenge.

Assumption: F a perfect field.

Recall: Voevodsky's triangulated category.

$DM_-^{eff}(F)$: Triangulated Category of bounded above effective motivic complexes.

$DM^{eff}(F)$: Unbounded version of the same.

PST = Category of presheaves with transfers
= Mod-SmCor(F) (Cov. functors SmCor(F) $\to$ Ab)

NST = Category of Nisnevich sheaves with transfers; full subcategory of PST formed by those presheaves which are sheaves in the Nisnevich topology.

$HI \subseteq NST$: Category of homotopy invariant Nisnevich sheaves with transfers.

$$(\mathcal{F}(X \times \mathbb{A}^1) \cong \mathcal{F}(X)).$$

$HI^{\circ} \subseteq PST$: full subcategory; consisting of those presheaves $\mathcal{F}$ that are birationally invariant; $\mathcal{F}(X) \simeq \mathcal{F}(U)$ for any open immersion $U \subset X$.

N.B: Any $\mathcal{F} \in PST$ which is birationally invariant is automatically a sheaf in the Nisnevich topology.

Recall also : SmCor(k) (Category of Correspondences)

Sm(k)

$$L : K^b(\mathrm{SmCor}) \longrightarrow D(PST) \quad \Big| \quad X \rightsquigarrow L(X)$$

X smooth proj. /k.

$L(X)$ NST.

Prop: One has $HI^{\circ} \subset HI$

Recap:

$$DM^{eff}(F) \longrightarrow DM^{\circ}(F)$$

- $DM^{eff}(F)$ equipped with a homotopy t-structure whose heart is $HI(F)$.

- $DM^{\circ}(F)$: Localisation of $D(PST)$; Triangulated category, heart $HI^{\circ}(F)$.

$$\begin{tikzcd} DM^{eff}(F) \arrow[r, shift left, "R_{nr}"] & DM^{\circ}(F) \arrow[l, shift left, "i^{\circ}"] \\ HI(F) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, phantom, "\supset"] & HI^{\circ}(F) \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

i° fully faithful.

$\mathcal{F} \in HI(F)$ is in $HI^{\circ}(F)$ if and only if it is locally constant in the Zariski topology.

$$\begin{tikzcd} DM^{eff}(F) \arrow[r, shift left=2, "R_{nr}"] \arrow[r, shift right=2, "\nu_{\leq 0}"'] & DM^{\circ}(F) \arrow[l, "i^{\circ}"'] \end{tikzcd}$$

- R_{nr} right adjoint to i°
- $\nu_{\leq 0}$ left adjoint to i°.

Defn : a) $R^{\circ}_{nr} = \mathcal{H}^{\circ} \circ R_{nr} : HI(F) \longrightarrow HI^{\circ}(F)$

(Both $HI(F)$ and $HI^{\circ}(F)$ are abelian Categories.

b) $\mathcal{F} \in HI(F)$; the contraction of $\mathcal{F}$, denoted $\mathcal{F}_{-1}$ is defined $\mathcal{F}_{-1} := \underline{Hom}_{HI}(\mathbb{G}_m, \mathcal{F})$. Here $\underline{Hom}$ is the internal Hom on HI; $\mathbb{G}_m$ the simplest example of a homotopy invariant element in NST.

$$\begin{aligned} HI(F) &\overset{i^{\circ}}{\hookrightarrow} HI^{\circ}(F) \\ \mathcal{F} &\longmapsto R^{\circ}_{nr}(\mathcal{F}) \end{aligned}$$

is right adjoint to i°

c) $\mathcal{F} \in HI(F)$, X a smooth, connected F-variety.

$$\mathcal{F}_{nr}(X) = \mathrm{Ker}\left(\mathcal{F}(K) \xrightarrow{\partial_v} \prod_v \mathcal{F}_{-1}(F(v))\right)$$

$K = F(X)$ function field

v varies over all divisorial discrete valuations on K, trivial on F, and $F(v)$ is residue field for v.

d) $\mathcal{F}_{nr} := R^{\circ}_{nr}\mathcal{F}$, "unramified part of $\mathcal{F}$."

<u>Connecting back</u>: Let $i \geq 0$, $j \in \mathbb{Z}$, n positive integer invertible in F.

Consider the Nisnevich sheaf $\mathcal{F} = \mathcal{H}^i_{et}(\mu_n^{\otimes j})$ associated to the presheaf

$$U \longmapsto H^i_{et}(U, \mu_n^{\otimes j})$$

defines an object of HI, and $R^0_{nr}\mathcal{F}$ is the usual unramified cohomology defined above.

Note that we now have an 'enhanced' object which contains more information! Namely, if $\mathcal{F} \in HI$, then $R_{nr}(\mathcal{F})$ is a complex, and we set:

$$R^q_{nr}(\mathcal{F}) := \mathcal{H}^q(R_{nr}(\mathcal{F}))$$

$$R^q_{nr}(\mathcal{F}) \in HI^o(F)!$$

We thus obtain lots of birational sheaves with transfers, associated with $\mathcal{F} \in HI$, and thus birational invariants. As explained above, unramified cohomology is R^0_{nr}!

§: Computations: Simplest examples: $\mathcal{F} = \mathbb{G}_m$ ("sheaf $\mathcal{K}_1$"), and $\mathcal{F} = \mathcal{K}_2$.

Caveats:

- Cannot explicitly compute beyond $q = 2$ for

- Can do better for varieties of dim ≤ 2, and this already yields interesting connections with other birational invariants.

◆◆ Further assumptions:

- (For simplicity): $F = \bar{F}$, algebraically closed.
- Cohomology will be NISNEVICH Cohomology.
- α: Projection of the étale site on smooth F-varieties onto the corresponding Nisnevich site.

X connected smooth projective F-variety.

- $\mathrm{Pic}^\tau(X)$ = Group of cycle classes in $\mathrm{Pic}(X) = CH^1(X)$ which are numerically equivalent to 0.

- $A_1^{alg}(X)$ = Group of 1-cycles ~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~ on X modulo algebraic equivalence.

- $N_1(X)$ = Group of 1-cycles on X modulo numerical equivalence.

- $N_1^1(X) = \dfrac{\mathrm{Pic}(X)}{\mathrm{Pic}^\tau(X)}$, finitely generated.

- $D^1(X) = \mathrm{Coker}\,(\underbrace{N^1(X) \longrightarrow \mathrm{Hom}(N_1(X), \mathbb{Z})}_{\text{induced by intersection pairing.}})$

(finite group)

- $NS_1(X, \tau)$: Groups introduced by Ayoub - Barbieri-Viale.

- $\mathrm{Griff}_1(X) := \mathrm{Ker}\,(A_1^{alg}(X) \longrightarrow N_1(X))$

Theorem 1: Let X be a connected, smooth, projective F-variety. Then:

(i) $R^0_{nr}\, \mathbb{G}_m(X) = F^*$

(ii) $R^1_{nr}\, \mathbb{G}_m(X) \simeq \mathrm{Pic}^{\tau}(X)$

(iii) There is a short exact sequence

$$0 \to D'(X) \to R^2_{nr}\, \mathbb{G}_m(X) \to \mathrm{Hom}(\mathrm{Griff}_1(X), \mathbb{Z}) \to 0$$

(iv) For $q \geq 3$, we have short exact sequences

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(NS_1(X, q-3), \mathbb{Z}) \to R^q_{nr}\, \mathbb{G}_m(X) \twoheadrightarrow \mathrm{Hom}_{\mathbb{Z}}(NS_1(X, q-2), \mathbb{Z})$$

For any smooth, projective X/F, put

$$\mathbb{G}_m^{\acute{e}t} := R\alpha_* \alpha^* \mathbb{G}_m \quad (\alpha \rightsquigarrow \text{change of site})$$

There is a natural map $\mathbb{G}_m \to \mathbb{G}_m^{\acute{e}t}$.

Can show that there is an exact sequence

$$0 \to R^2_{nr}\, \mathbb{G}_m(X) \to R^2_{nr}\, \mathbb{G}_m^{\acute{e}t}(X) \to Br(X).$$

$\mathcal{F} = \mathcal{K}_2$:

Theorem: For any smooth projective variety X/F, we have an exact sequence

$$0 \to \widetilde{\mathrm{Pic}}(X)F^* \to (R^1_{nr}\mathcal{K}_2)(X) \to H^1_{ind}(X, \mathcal{K}_2)$$

$$\to \mathrm{Hom}(\mathrm{Griff}(X), F^*) \to (R^2_{nr}\mathcal{K}_2)(X) \to CH^2(X)$$

Here:

- $H^1_{ind}(X, \mathcal{K}_2) = \mathrm{Coker}(\mathrm{Pic}(X) \otimes F^* \to H^1(X, \mathcal{K}_2))$.
- $\widetilde{\mathrm{Pic}}(X)F^* = \mathrm{Im}(\widetilde{\mathrm{Pic}}(X) \otimes F^* \to H^1(X, \mathcal{K}_2))$.

Thank you for your attention!

Vive la Coopération franco-Indienne