Connected Dominating Set in Graphs with (no) Small Cycles

Fixed-Parameter Tractability
Kernelization Complexity

N Misra, G Philip, V Raman, and S Saurabh

The Institute of Mathematical Sciences, Chennai, India

TU Berlin
Berlin, Germany
November 3, 2010

Outline

Outline

Dominating Set

Definition (Dominating Set)

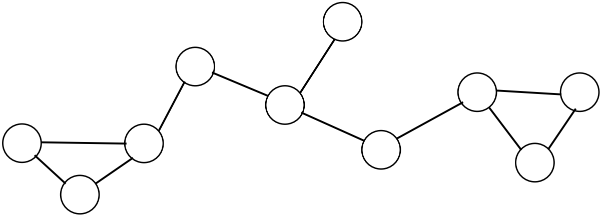
Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *dominating set* of G if every vertex which is not in S has at least one neighbour in S.

Dominating Set

Definition (Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *dominating set* of G if every vertex which is not in S has at least one neighbour in S.

Example

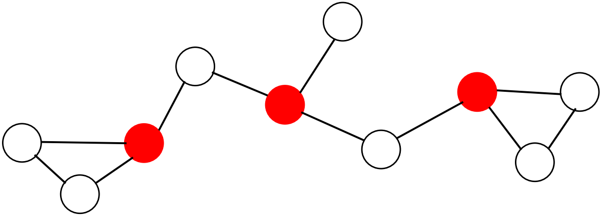


Dominating Set

Definition (Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *dominating set* of G if every vertex which is not in S has at least one neighbour in S.

Example



Connected Dominating Set

Definition (Connected Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *connected* dominating set of G if

- S is a dominating set of G, and,
- the subgraph $G[S]$ induced by S is connected.

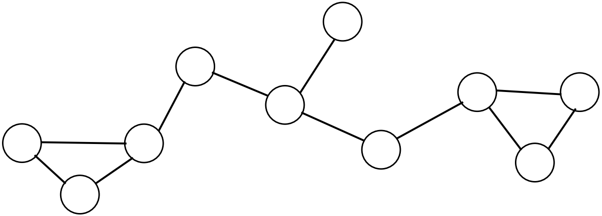
Connected Dominating Set

Definition (Connected Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *connected* dominating set of G if

- S is a dominating set of G, and,
- the subgraph $G[S]$ induced by S is connected.

Example



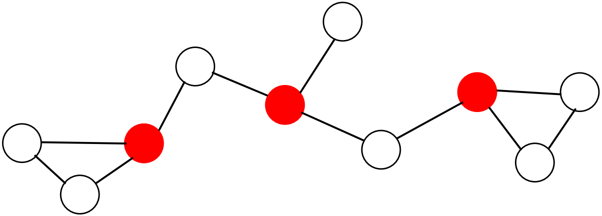
Connected Dominating Set

Definition (Connected Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *connected* dominating set of G if

- S is a dominating set of G, and,
- the subgraph $G[S]$ induced by S is connected.

Example



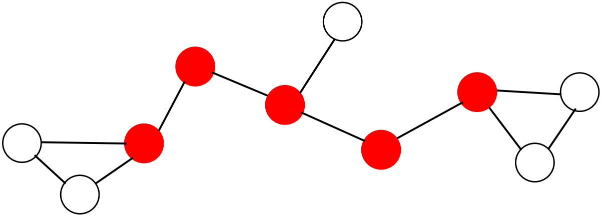
Connected Dominating Set

Definition (Connected Dominating Set)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices of G is said to be a *connected* dominating set of G if

- S is a dominating set of G, and,
- the subgraph $G[S]$ induced by S is connected.

Example



The CDS problem

Definition (The CDS problem)

- Input: A graph G and a positive integer k.
- Question: Does G have a connected dominating set of size at most k?

Classical Complexity

- The problem is NP-complete, even in planar graphs.
- The best known polynomial-time approximation factor is $\ln(\Delta) + 3$.

The CDS problem

Definition (The CDS problem)

- Input: A graph G and a positive integer k.
- Question: Does G have a connected dominating set of size at most k?

Classical Complexity

- The problem is NP-complete, even in planar graphs.
- The best known polynomial-time approximation factor is $\ln(\Delta) + 3$.

Outline

PC: Brief Overview 1/3

- Goal: Solve NP-hard problems in polynomial time …
  - …when some aspect (the *parameter*) of the input is bounded.
- A parameterized problem instance has two parts:
  - The input itself, and,
  - A *parameter*, usually a number denoted by k.

PC: Brief Overview 1/3

- Goal: Solve NP-hard problems in polynomial time ...
  - ... when some aspect (the *parameter*) of the input is bounded.
- A parameterized problem instance has two parts:
  - The input itself, and,
  - A *parameter*, usually a number denoted by k.

PC: Brief Overview 2/3

- A parameterized problem is *fixed-parameter tractable* (**FPT**) if
  - It can be solved in time $f(k) \cdot n^c$ time, where
  - c is a constant, $f()$ depends only on k.
- A *kernelization algorithm* for a parameterized problem
  - Runs in time polynomial in the input size, and
  - Outputs an **equivalent** instance — a *kernel* — of size bounded by $g(k)$.
- Fact: A parameterized problem is FPT if and only if it has a kernelization algorithm.

PC: Brief Overview 2/3

- A parameterized problem is *fixed-parameter tractable* (**FPT**) if
  - It can be solved in time $f(k) \cdot n^c$ time, where
  - c is a constant, $f()$ depends only on k.
- A *kernelization algorithm* for a parameterized problem
  - Runs in time polynomial in the input size, and
  - Outputs an **equivalent** instance — a *kernel* — of size bounded by $g(k)$.
- Fact: A parameterized problem is FPT if and only if it has a kernelization algorithm.

PC: Brief Overview 3/3

Given a parameterization of an NP-hard problem, we ask:

- Is the problem FPT?
  - Does it have an algorithm that runs in $f(k) \cdot n^c$ time?
  - There is a hardness theory to show that certain problems are *unlikely* to be FPT.
- If YES, does it have a small kernel?
  - Polynomial-size kernels are good, linear-size even better.
  - There is a hardness theory to show that certain problems are *unlikely* to have polynomial-size kernels.

PC: Brief Overview 3/3

Given a parameterization of an NP-hard problem, we ask:

- Is the problem FPT?
  - Does it have an algorithm that runs in $f(k) \cdot n^c$ time?
  - There is a hardness theory to show that certain problems are *unlikely* to be FPT.
- If YES, does it have a small kernel?
  - Polynomial-size kernels are good, linear-size even better.
  - There is a hardness theory to show that certain problems are *unlikely* to have polynomial-size kernels.

The CDS problem

Parameterized Version

- Input: A graph G and a positive integer k.
- Parameter: k
- Question: Does G have a connected dominating set of size at most k?

The CDS problem

Parameterized Complexity: Known Results

- W[2]-hard on graphs in general:
    - $O(f(k) \cdot poly(n))$-time algorithms are unlikely to exist.
- FPT on graphs of bounded degeneracy (Golovach, Villanger : WG 2008).
    - Planar graphs, graphs of bounded treewidth, ...
- Has a *linear* kernel in apex-minor-free graphs (Fomin, Lokshtanov, Saurabh, Thilikos : SODA 2010).
- *Unlikely* to have polynomial kernels in graphs of bounded degeneracy (Cygan, Pilipczuk, Pilipczuk, Wojtaszczyk : WG 2010).

Outline

The CDS problem

Parameterized Complexity: Our Results

- We explore what happens when the *girth* of the input graph is bounded.
    - The girth of a graph G is the size (number of edges) of a smallest cycle in G.
    - "Orthogonal" to degeneracy:
        - There are graphs of fixed girth and arbitrarily large degeneracy, and vice versa.

The CDS problem

Parameterized Complexity: Our Results

- We completely characterize the parameterized complexity of CDS for different values of girth:
  - W[2]-hard on graphs of girth 3 or 4.
  - FPT on graphs of girth at least 5.
  - Polynomial kernels unlikely in graphs of girth 5 or 6.
  - Cubic ($O(k^3)$) kernel in graphs of girth at least 7.

Outline

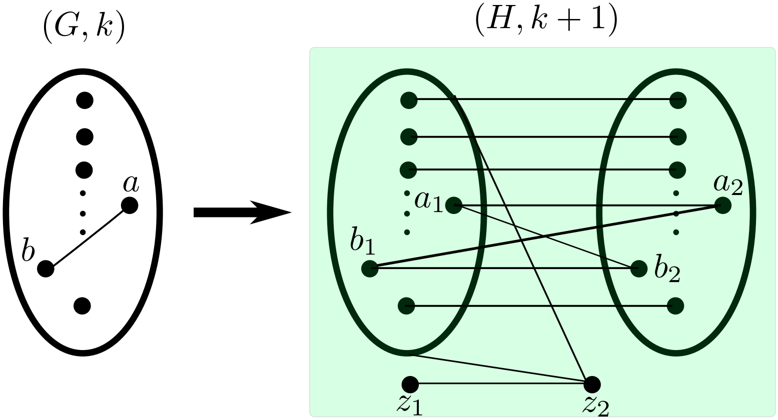
CDS in Graphs of Girth 3 or 4

W[2]-hardness Reduction

- Reduction from Dominating Set.
- Appears in earlier work of Raman and Saurabh.
- We merely observe that it works for CDS as well ... ☺

CDS in Graphs of Girth 3 or 4

W[2]-hardness Reduction



Outline

CDS in Graphs of Girth at least 5

FPT Algorithm: Outline

- The girth bound implies a degree-rule:
  - Every vertex of "high" degree must be in *any* small CDS (in fact, of any small *DS*).
- If there are too many high-degree vertices, then say NO.
- If the number of **undominated** vertices is too much, then say NO.
- The number of vertices that could possibly be part of a solution is bounded:
  - Guess a minimal DS from among these
  - Find a minimum-size Steiner tree for this DS.

CDS in Graphs of Girth at least 5

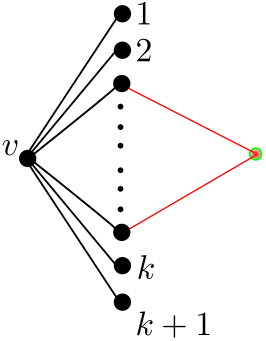
FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.

CDS in Graphs of Girth at least 5

FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.



CDS in Graphs of Girth at least 5

FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.
- We use colours for keeping track of the state.
  - Red vertices are in the dominating set that we construct.
  - Vertices dominated by the red ones are green.
  - Undominated vertices are blue.
- Reduction Rule 0: Colour all vertices blue.

CDS in Graphs of Girth at least 5

FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.
- Reduction Rule 1: If a vertex v has degree more than k:
  - Colour v red.
  - Colour all neighbours of v green.

CDS in Graphs of Girth at least 5

FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.
- Reduction Rule 1: If a vertex v has degree more than k:
  - Colour v red.
  - Colour all neighbours of v green.
- Reduction Rule 2: If there are more than $k(k+1)$ blue vertices left, say **NO**:
  - k vertices of degree at most k can dominate at most $k(k+1)$ blue vertices.

CDS in Graphs of Girth at least 5

FPT Algorithm: The Degree Rule

- If a vertex v has degree more than k, then v must be in every dominating set of size at most k.
- Reduction Rule 1: If a vertex v has degree more than k:
  - Colour v red.
  - Colour all neighbours of v green.
- Reduction Rule 2: If there are more than $k(k+1)$ blue vertices left, say **NO**:
- Reduction Rule 3: If there are more than k red vertices, say **NO**.

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.
- Every inclusion-minimal **dominating set** of size at most k is contained in these $k + k(k+1) + k^2(k+1)$ vertices, say $\mathcal{S}$.

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.

- Every inclusion-minimal dominating set of size at most k is contained in these $k + k(k+1) + k^2(k+1)$ vertices, say $\mathcal{S}$.
- Go over all minimal dominating sets $D \subseteq \mathcal{S}$ of size at most k.

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.
- Every inclusion-minimal dominating set of size at most k is contained in these $k + k(k+1) + k^2(k+1)$ vertices, say $\mathcal{S}$.
- Go over all minimal dominating sets $D \subseteq \mathcal{S}$ of size at most k.
  - This can be done in $k^{O(k)}$ time.

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.

- Every inclusion-minimal dominating set of size at most k is contained in these $k + k(k+1) + k^2(k+1)$ vertices, say $\mathcal{S}$.
- Go over all minimal dominating sets $D \subseteq \mathcal{S}$ of size at most k.
- For each D, check if there is a Steiner tree on at most k vertices connecting D.
  - This can be done in $2^{O(k)}$ time (Nederlof, ICALP 2009).

CDS in Graphs of Girth at least 5

The FPT Algorithm

- There are at most
  - k red vertices.
  - $k(k+1)$ blue vertices.
  - $k^2(k+1)$ *neighbours* of blue vertices.

- Every inclusion-minimal dominating set of size at most k is contained in these $k + k(k+1) + k^2(k+1)$ vertices, say $\mathcal{S}$.
- Go over all minimal dominating sets $D \subseteq \mathcal{S}$ of size at most k.
- For each D, check if there is a Steiner tree on at most k vertices connecting D.
- A CDS of the graph, of size at most k, if it exists, can thus be found in $O^*(k^{O(k)})$ time.

Outline

The Connected Dominating Set problem

- Introduction
- Parameterized Complexity
- Our Results

CDS and Girth

- Graphs of Girth 3 or 4 : W-hard
- Graphs of Girth at least 5 : FPT
- Graphs of Girth 5 or 6 : No Polynomial Kernels
- Graphs of Girth at least 7 : Cubic Kernel

Kernel Lower Bound Machinery

- We make use of kernel lower bound techniques developed by
  - Bodlaender, Downey, Fellows, Hermelin : *JCSS* 2009
  - Bodlaender, Thomassé, Yeo : *ESA* 2009
- These techniques allow us to conclude that a problem does not have polynomial kernels ...
  - *unless* $NP \subseteq CoNP/Poly$
  - or (weaker) PH collapses to the third level

Kernel Lower Bound Machinery

- We make use of kernel lower bound techniques developed by
  - Bodlaender, Downey, Fellows, Hermelin : *JCSS* 2009
  - Bodlaender, Thomassé, Yeo : *ESA* 2009
- These techniques allow us to conclude that a problem does not have polynomial kernels ...
  - *unless* $NP \subseteq CoNP/Poly$
  - or (weaker) PH collapses to the third level

Composition Algorithm

A composition algorithm for a parameterized problem

- takes as input a *sequence* $\langle (x_1, k), (x_2, k), \ldots, (x_t, k) \rangle$
- runs in time polynomial in $\sum_{i=1}^{t} |x_i| + k$
- outputs an instance (y, k') :
    - (y, k') is YES iff **at least one** (x_i, k) is YES,
    - k' is polynomial in k.

The composition algorithm acts like an **OR** gate.

Kernel Lower Bound Machinery

Tools : Composition

Theorem

If a parameterized NP-complete problem has a composition algorithm, then it does not have polynomial kernels

... unless $NP \subseteq CoNP/Poly$.

Kernel Lower Bound Machinery

Tools : Composition

Theorem

If a parameterized NP-complete problem has a composition algorithm, then it does not have polynomial kernels

. . . unless $NP \subseteq CoNP/Poly$.

Kernel Lower Bound Machinery

Tools : Reduction

- Let $\mathcal{A}, \mathcal{B}$ be parameterized problems.

Definition (Polynomial Parameter Transformation)

A *polynomial parameter transformation* (**PPT**) from $\mathcal{A}$ to $\mathcal{B}$ is a function f such that:

- f is polynomial-time computable.
- If $f(x, k) = (x', k')$, then
  - $(x, k) \in \mathcal{A} \iff (x', k') \in \mathcal{B}$.
  - $k' \leq p(k)$ for some polynomial p.

Kernel Lower Bound Machinery

Tools : Reduction

Theorem

Let $\mathcal{A}, \mathcal{B}$ be parameterized problems where $\mathcal{A}$ is NP-complete and $\mathcal{B}$ is in NP. Suppose there is a PPT from $\mathcal{A}$ to $\mathcal{B}$. Then, if $\mathcal{B}$ has a polynomial kernel, then $\mathcal{A}$ also has a polynomial kernel.

- Composition: **OR**-like algorithm, implies no poly-kernel unless . . .
- Reduction: In polynomial time, and at most polynomial increase in the parameter. Helps propagate no-poly-kernel bounds.

CDS in Graphs of Girth 5 or 6

Kernel Lower Bound

- We could not devise a composition algorithm
  - A composition algorithm is sometimes *very* hard to find
  - . . . and may be it doesn't exist (we don't know).
- We invented an intermediate problem, Fair Connected Colours (FCC).
  - It's a snap to compose FCC.
- We found a PPT (reduction) from FCC to CDS in graphs of girth 5 or 6.
  - Not easy either, but it could be done ☺.

CDS in Graphs of Girth 5 or 6

Kernel Lower Bound

- We could not devise a composition algorithm
  - A composition algorithm is sometimes *very* hard to find
  - . . . and may be it doesn't exist (we don't know).
- We invented an intermediate problem, Fair Connected Colours (FCC).
  - It's a snap to compose FCC.
- We found a PPT (reduction) from FCC to CDS in graphs of girth 5 or 6.
  - Not easy either, but it could be done ☺.

CDS in Graphs of Girth 5 or 6

Kernel Lower Bound

- We could not devise a composition algorithm
  - A composition algorithm is sometimes *very* hard to find
  - ...and may be it doesn't exist (we don't know).
- We invented an intermediate problem, Fair Connected Colours (FCC).
  - It's a snap to compose FCC.
- We found a PPT (reduction) from FCC to CDS in graphs of girth 5 or 6.
  - Not easy either, but it could be done ☺.

CDS in Graphs of Girth 5 or 6

Fair Connected Colours (FCC)

- Input: A graph G whose vertices are *properly* coloured with k colours such that all neighbours of each vertex have distinct colours.
- Parameter: k
- Question: Does G contain a tree T on k vertices (as a subgraph) where all vertices in T have different colours?

CDS in Graphs of Girth 5 or 6

Fair Connected Colours (FCC)

- FCC is NP-complete
  - Reduction from SAT
- FCC is compositional
  - Just take disjoint union!
- So from the earlier theorem, we have:
  - FCC has no polynomial kernels unless . . .

CDS in Graphs of Girth 5 or 6

Fair Connected Colours (FCC)

- FCC is NP-complete
  - Reduction from SAT
- FCC is compositional
  - Just take disjoint union!
- So from the earlier theorem, we have:
  - FCC has no polynomial kernels unless ...

CDS in Graphs of Girth 5 or 6

Fair Connected Colours (FCC)

- FCC is NP-complete
    - Reduction from SAT
- FCC is compositional
    - Just take disjoint union!
- So from the earlier theorem, we have:
    - FCC has no polynomial kernels unless . . .

CDS in Graphs of Girth 5 or 6

Reduction from FCC : Rough Picture

- Given an instance (G, k) of FCC, we construct an instance $(H, k^2 + k)$ of CDS such that:
  - H has girth 6.
  - A colourful tree in G becomes a connected dominating set in H:
    - The one vertex in each colour class dominates (well, sort of) all of that class.
    - The underlying tree provides connectivity.
  - And vice versa.
    - The $+k^2$ is spent for increasing the girth, and for domination.
  - The modification for girth 5 is not difficult.

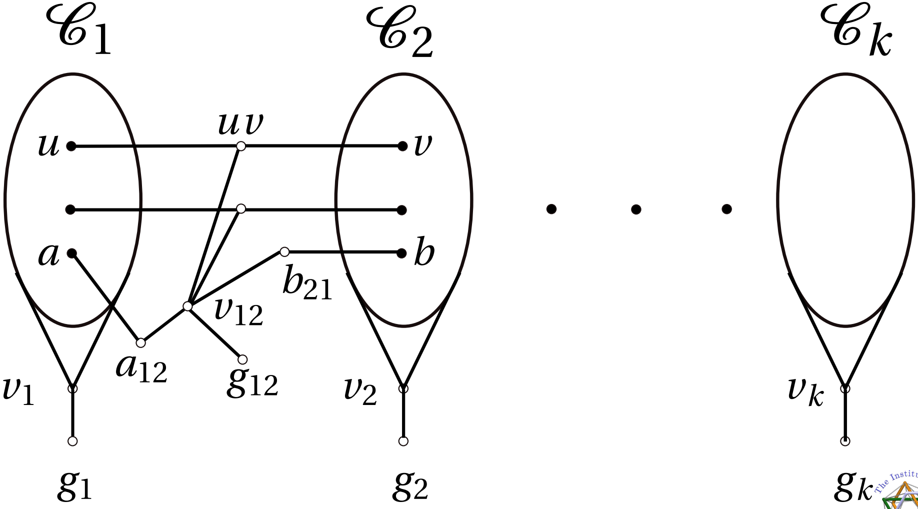
CDS in Graphs of Girth 5 or 6

Reduction from FCC : Rough Picture

- Given an instance (G, k) of FCC, we construct an instance $(H, k^2 + k)$ of CDS such that:
  - H has girth 6.
  - A colourful tree in G becomes a connected dominating set in H:
    - The one vertex in each colour class dominates (well, sort of) all of that class.
    - The underlying tree provides connectivity.
  - And vice versa.
    - The $+k^2$ is spent for increasing the girth, and for domination.
  - The modification for girth 5 is not difficult.

CDS in Graphs of Girth 5 or 6

Reduction from FCC : ~~Rough~~ Picture



CDS in Graphs of Girth 5 or 6

Summary

- Fair Connected Colours is NP-complete and compositional.
- Fair Conneced Colours reduces to Connected Dominating Set
  - In polynomial time, and with polynomial increase in the parameter.
- These together imply that Connected Dominating Set is not likely to have polynomial kernels.

Outline

CDS in Graphs of Girth at Least 7

Cubic Kernel: Outline

- The main reduction rule is the same degree rule as before.
- As before, we use colours for keeping track of the state:
  - Red vertices are in the dominating set that we construct.
  - Vertices dominated by the red ones are green.
  - Undominated vertices are blue.
- The larger girth (7) allows us to bound the number of green vertices as well.
  - In girth-5 graphs, we could only bound the number of green vertices *which were adjacent to* blue vertices.

CDS in Graphs of Girth at Least 7

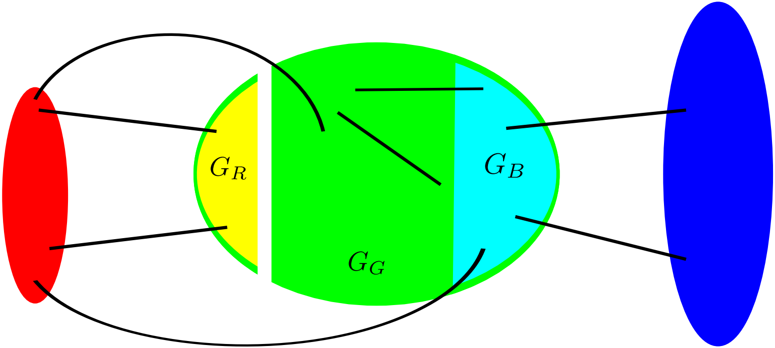
Cubic Kernel: Reduction Rules

- Reduction Rule 0: Colour all vertices blue.
- Reduction Rule 1: If $deg(v) > k$, colour v red and all neighbours of v green.
- Reduction Rule 2: If there are more than k red vertices or more than $k(k+1)$ blue vertices left, say **NO**.
- Reduction Rule 3: Say **NO** if there is an isolated blue vertex.
- Reduction Rule 4: If u is a pendant blue or green vertex adjacent to a vertex v:
  - Remove u.
  - Colour v red.
  - Colour all blue neighbours of v green.

CDS in Graphs of Girth at Least 7

Cubic Kernel: Bounding Green Vertices, Part 1

- Divide green vertices into three:
  - G_B : has *at least one* blue neighbour.
  - G_R : has *only* red neighbours.
  - G_G : no blue neighbour, at least one green neighbour.



CDS in Graphs of Girth at Least 7

Cubic Kernel: Bounding Green Vertices, Part 1

- Divide green vertices into three:
  - G_B : has *at least one* blue neighbour.
  - G_R : has *only* red neighbours.
  - G_G : no blue neighbour, at least one green neighbour.

- $|G_B| \leq k \cdot |B| \leq k(k^2 + k)$
  - The degree rule.
- $|G_R| \leq \binom{|R|}{2} \leq \binom{k}{2}$
  - The pendant rule, and no two vertices have more than one common neighbour.

CDS in Graphs of Girth at Least 7

Cubic Kernel: Bounding Green Vertices, Part 2

- Bounding $|G_G|$
  - Green vertices which have green neighbours.
  - Suffices to bound the number of *edges* E_G with *both* end points green.
    - $|G_G| \leq 2 \cdot |E_G|$
  - Each green vertex is adjacent to some red vertex.
  - For any pair x, y of red vertices, there is at most one edge $\{u, v\} \in E_G$.
    - Otherwise there is a 6-cycle!
  - $|G_G| \leq 2 \cdot |E_G| \leq 2 \cdot \binom{|R|}{2} \leq 2 \cdot \binom{k}{2}$

CDS in Graphs of Girth at Least 7

Cubic Kernel: Summary

- At most k red vertices.
- At most $k(k+1)$ blue vertices.
- At most $k(k^2+k)+3\cdot\binom{k}{2}$ green vertices.
- A coloured kernel on at most $k^3+\frac{7}{2}k^2+\frac{k}{2}$ vertices.
- To get a "plain" kernel:
    - Attach a new pendant vertex to each red vertex.
    - Remove all colours.
    - A kernel for CDS on at most $k^3+\frac{7}{2}k^2+\frac{3}{2}k$ vertices.

Recapitulating . . .

- Connected Dominating Set parameterized by solution size k is:
  - W[2]-hard on graphs in general.
  - W[2]-hard on graphs of girth at most 4.
  - FPT on graphs of girth at least 5.
  - No polynomial kernel (. . .) in graphs of girth 5 or 6.
  - A cubic ($O(k^3)$) kernel in graphs of girth at least 7.

Thank You!