Nature of Transition in Explosive Percolation

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Percolation

- Phase Transition between ordered and disordered phases.
- Correlation length diverges across a critical link / site density.
- Example: Random Resistor Network, Binary Alloys, Forest fires etc.
- Ordinary percolation and its many different variants exhibit Continuous Transitions.
- Kasteleyn and Fortuin: $q \to 1$ state Potts model is equivalent to Bond Percolation on square lattice.

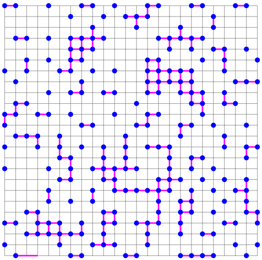


Figure: p=0.4

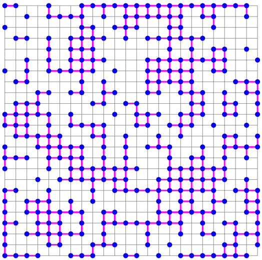


Figure: p=0.6

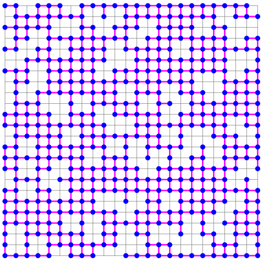


Figure: p=0.8

$p_c = 0.59274621(13)$ M. E. J. Newman and R. M. Ziff, Phys. Rev. Lett. **85**, 4104 (2000).

First Order Transition in Bootstrap Percolation

- A First Order Transition (FOT) is characterized by an abrupt jump in the Order Parameter.
- FOT has been observed in Bootstrap Percolation (BP) in some lattices [1].
- BP models competition between exchange and crystal field interactions in some magnetic materials: $Tb_c Y_{1-c} Sb$.
- Sites occupied (vacated) with probability p $(1-p)$.
- Culling: Occupied sites having < m occupied neighbors are recursively removed.

  For $p > p_c^*(m)$ the stable configuration has a spanning "infinite" cluster;
- For $m = 3$ (i) $p > p_c^*(3)$ the system has single spanning cluster, where as for $p < p_c^*(3)$ the lattice is completely empty.

BP config. at p_c

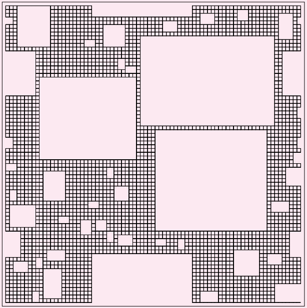
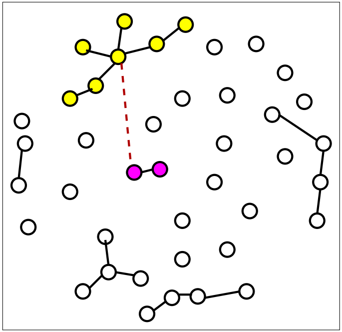


Figure: BP on a square lattice of size $L = 80$, for $m = 3$ and at the percolation threshold. $p_c^*(3) \approx 0.925$.

[1] J. Chalupa, P. L. Leath and G. R. Reich, J. Phys. C., **12**, L31 (1979).

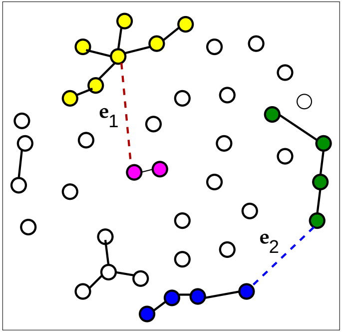
Erdös-Rényi (ER) Random Graph

- Starts with N isolated vertices and no edges.
- Edges added one by one; time $t = rN = \#$ of edges.
- Two vertices are randomly picked and are connected.
- Size s of a component = $\#$ of vertices connected by edges.
- Prob. that an edge connects 1 & 2 is $\propto s_1 s_2$.
- At $r < 1/2$, largest component size $\mathcal{C}_N \sim \ln(N)$.
- At $r > 1/2$, $\mathcal{C}_N \propto N$, and for $r \to 1/2+$, $\mathcal{C}_N \approx (4r-2)N$.
- A Continuous Phase Transition at $r = 1/2$ from local to global connectivity.



Achlioptas Process (AP)

- Two distinct edges e_1, e_2 are selected randomly.
- Example: e_1 between $s_1 = 7$ and $s_2 = 2$; e_2 between $s_3 = 4$ and $s_4 = 4$.
- Product Rule (PR): Select the link with smaller product: Since $s_1 s_2 < s_3 s_4$ occupy e_1, connect s_1 and s_2.
- Sum Rule (SR): Retain the link that minimizes the sum: Since $s_1 + s_2 > s_3 + s_4$ occupy e_2, connect s_3 and s_4.
- Small clusters grow faster, large clusters grow slower; as a result transition is delayed.
- Accelerated transition: occupy the edge with larger product (sum).



[1] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science **323**, 1453 (2009).

Order of Transition

- Order parameter $\mathcal{C}_N = s_{max}(N)/N$, $s_{max}(N)$ = Size of the largest component.
- t_0 is last step for $s_{max}(N) < N^{1/2}$ and t_1 is first step for $s_{max}(N) > N/2$.
- Time gap $\Delta = t_1 - t_0 \propto N$ for cont. trans., e.g., $\Delta > 0.193N$ for ER.
- Δ is not extensive for EP since $\Delta/N^{2/3} \to 1$.
- $\mathcal{C}_N$ jumps abruptly and instantaneously, similar to an explosion at $r_c \approx 0.888449(2)$.
- Hence the name: EXPLOSIVE PERCOLATION

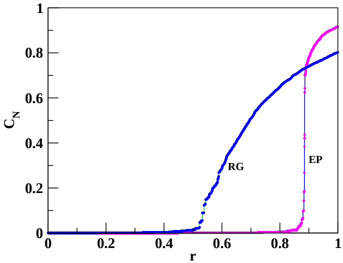
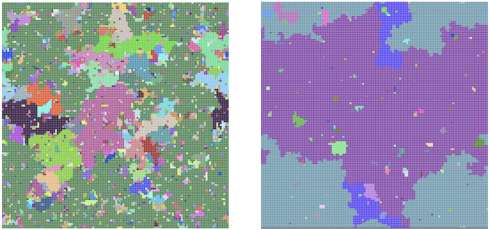


Figure: Variation of $\mathcal{C}_N$ with link density r for $N = 2^{14}$. Single configuration.

Percolation / Explosive Percolation on Square Lattice



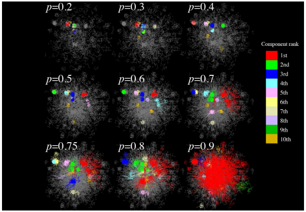
Left: Ordinary bond percolation; Right: Explosive Percolation.

R. M. Ziff, Phys. Rev. Lett. **103**, 045701 (2009).

Human Protein Homology Network

- Similarity exists between proteins of human cells. They are homologous.
- Degree of homology between different pairs of proteins are different.
- One defines a weighted network depending on the degree of homology. Proteins form modules.
- Initially all proteins and no links. One starts with largest weight link, adds one link at a time with decreasing order of the weight.
- Largest cluster very small even with 80% of links But increases sharply beyond that like EP.

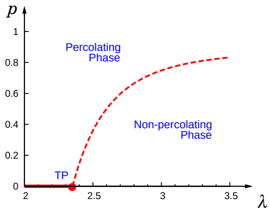
H. D. Rozenfeld, L. K. Gallos and H. A. Makse, Eur. Phys. J. B **75**, 305 (2010).



- Snapshots of H-PHN network at different link density p.

Explosive Percolation in Scale-Free networks

- Chung and Lu (CL) model of Scale-Free Network.
- N nodes, $i = 1, 2, 3, \ldots, N$, each has weight $w_i = (i + i_0 - 1)^{-\mu}$ where $i_0 \propto N^{1-1/2\mu}$ for $1/2 < \mu < 1$ and $i_0 = 1$ for $\mu < 1/2$.
- Nodes (i, j) selected with probs. $w_i/\Sigma_k w_k$ and $w_j/\Sigma_k w_k$ are connected.
- One gets a scale-free network with degree dist. $P(k) \sim k^{-\lambda}$ with $\lambda = 1 + 1/\mu$.
- Now apply PR rule of AP.
- Randomly select edge pairs (i, j) by $w_i w_j/(\Sigma_k w_k)^2$ and (l, m) by $w_l w_m/(\Sigma_k w_k)^2$.
- The edge with smaller value of the product of component sizes is occupied.
- Result: As $N \to \infty$, $p_c = 0$ for $\lambda < \lambda_c$ where $2.3 < \lambda_c < 2.4$, jump in $\mathcal{C}$ is also zero.
- For $\lambda > \lambda_c$ $p_c > 0$ and $\Delta/N^{0.8}$ = constant > 0, indicating FOT.



- CT (solid line), FOT (dashed line).

[1] Y.S. Cho, J. S. Kim, J. Park, B. Kahng and D. Kim, Phys. Rev. Lett. **103**, 135702 (2009).

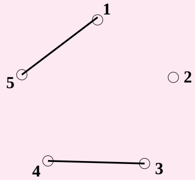
Why first order transition in EP?

- FOT is like a catastrophic process, a small change in control variable leads to a spurt of activity on a global scale.
- Occupation rule in EP is biased.
- Discourages large clusters to grow.
- Encourages faster growth of smaller clusters.
- Clusters should have comparable sizes.
- Nearly equal size clusters are then linked together by additional few edges.
- This results abrupt global connection as in the 'First Order Transition' (FOT).

A new route to Explosive Percolation

- A single edge with prob. π_{ij} from the whole set of vacant edges is selected and occupied.
- The prob. $\pi_{ij} \propto (s_i s_j)^\zeta$ is continuously tuned by a parameter ζ.
- For $\zeta < 0$, edges with smaller products are preferred.
- For $\zeta = 0$ it is Random Graph.
- For $\zeta > 0$, edges with larger products are preferred.
- Numerics indicate for $\zeta < \zeta_c$ is a FOT.
- $\zeta_c = -1/2$ for RG and 0 for SL.

Example: A graph of 5 nodes

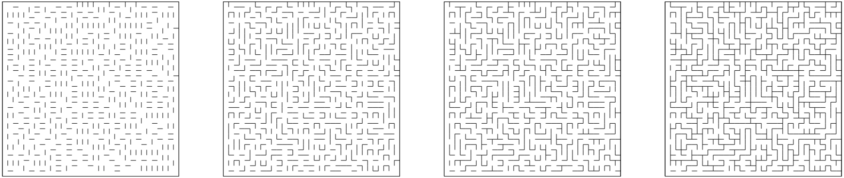


- 3 components, 8 vacant edges.
- Assign weights: $w_{12} = (2.1)^\zeta$, $w_{13} = (2.2)^\zeta$, $w_{14} = (2.2)^\zeta$, $w_{23} = (1.2)^\zeta$, $w_{24} = (1.2)^\zeta$, $w_{25} = (1.2)^\zeta$, $w_{35} = (2.2)^\zeta$, $w_{45} = (2.2)^\zeta$.
- $W = \Sigma w_{ij} = w_{12}+w_{13}+w_{14}+w_{23}+w_{24}+w_{25}+w_{35}+w_{45}$
- Occupation probabilities are $\pi_{ij} = w_{ij}/W$.

• S. S. Manna and A. Chatterjee, Physica A, **390**, 177 (2011).

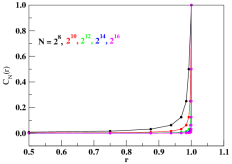
Limit of $\zeta \to -\infty$ for SL

- A series of jammed states. For $\zeta \to -\infty$ only one edge is randomly selected from the subset of vacant edges whose $s_i s_j$ are minimum and is occupied.
- When this subset is exhausted (jammed), $(s_i s_j)_{min}$ is increased and a new subset of vacant edges is obtained.
- Start with $N = L^2$ isolated nodes and no edge, i.e., $(s_i s_j)_{min} = 1$.
- Overlap of two occupied edges forbidden.
- Jammed state has only single bonds and isolated nodes.
- Random Sequential Adsorption (RSA).
- Jamming densities are 0.227, 0.273, 0.351, 0.370 .... approaches $p_c = 1/2$ which is a Spanning Tree config.



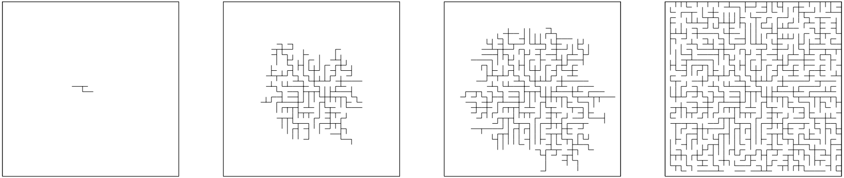
Limit of $\zeta \to -\infty$ for RG

- Cluster sizes are in GP
- Minimal products are 1, 4, 16, 64, 256, ... etc.
- Cluster sizes are 1, 2, 4, 8, 16, ... , $N/2$ and N.
- Jamming densities are 1/2, 3/4, 7/8, ... and $(s-1)/s$ in general. This implies $r_c = 1$.
- The largest cluster size when plotted with link density gives a step function at $r = 1$, i.e., a perfect first order transition.



Limit of $\zeta \to +\infty$ for SL

- Vacant edges with maximum value of $s_i s_j$ are only occupied.
- Once first edge occupied, other edges are connected to this edge.
- Growth is limited to surface bonds leading to:
- Loop-less growth: Bond Eden Tree (BET) [1]



[1] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **54**, 1346 (1985), S. S. Manna and D. Dhar, Phys. Rev. E. **54**, R3063 (1996).

Limit of $\zeta \to +\infty$ for RG

Random network

- A single cluster growth process.
- A new node is connected to a randomly selected node of the growing cluster.
- This is the model A network of [1].
- This network has exponentially decaying degree distribution.

[1] A.-L. Barabási and R. Albert, Science, **286**, 509 (1999).

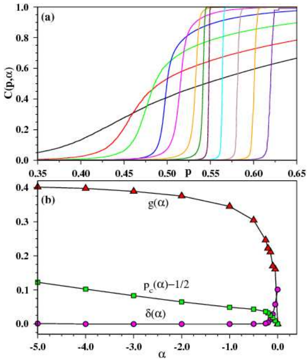
ζ finite; Different Measures

- Let $r_{max}(N)$ be the link density where the largest jump occurs in $\mathcal{C}_N$.
- We define $r_c(N) = \langle r_{max}(N) \rangle$ as the critical link density or, the percolation threshold.

- How much $\mathcal{C}_N$ increases due to a single link addition?
- $\langle \Delta\mathcal{C}_{max}(N) \rangle$ = av. value of maximal jump in $\mathcal{C}_N$ due to single link addition.
- Define $g(N) = \langle \Delta\mathcal{C}_{max}(N) \rangle = g + AN^{-\eta}$ such that $g = \lim_{N\to\infty} g(N)$. $g = 0$ corresponds to continuous transition and $g > 0$ signifies discontinuous transition.
- For our model we get $g(\zeta)$ for a given value of ζ.

- Define: t_0 is the latest time with $s_{max} < N^{1/2}$ and t_1 is the earliest time $s_{max} > N/2$.
- A gap $\Delta = t_1 - t_0$ in $N \to \infty$ limit can distinguish between two kinds of transitions: $\delta(\zeta) = \lim_{N\to\infty} \frac{\Delta(\zeta,N)}{N} = constant > 0$ for a continuous transition and 0 for a FOT.

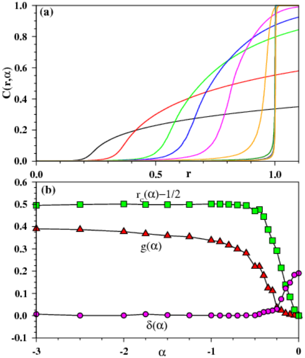
- Percolation with loops.
- For $-5 \leq \zeta \leq -1$ curves are nearly vertical.
- For $-1 < \zeta < 0$, the curves are relatively smooth but gradually become steeper as N increases.
- $p_c(\zeta)$ increases with decreasing ζ; reaches 1 as $\zeta \to -\infty$.
- $g(\zeta)$ jumps to 0.16 at $\zeta = -0.05$ and then gradually increases to 1/2 as $\zeta \to -\infty$.
- $\delta(\zeta)$ values are nearly zero for $-5 \leq \zeta \leq -1/4$, indicating FOT.
- Conclusion: $\zeta_c = 0$ for our model on SL.



- Order parameter $\mathcal{C}(p, \zeta)$ with link density p for ζ = 1/2, 0.2, 0.1, 0, -0.1, -1/4, -1/2, -1, -2, -3, -4 and -5, ζ values decreasing from left to right, for SL of $L = 512$.
- The asymptotic values of the gap $\delta(\zeta)$, the percolation threshold $p_c(\zeta) - 1/2$ and the largest jump $g(\zeta)$ of the Order Parameter plotted with ζ.

ζ finite; Random Graph

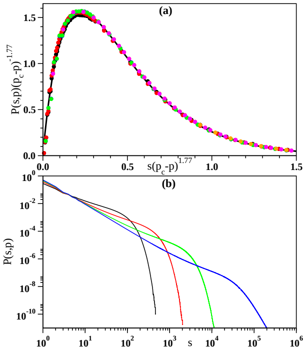
- $\mathcal{C}(r, \zeta)$ becomes increasingly steeper as ζ decreases from 0.
- for $\zeta \leq -1$ the curves almost coincide.
- $r_c(\zeta)$ saturates to 1 for $\zeta < -1/2$.
- $g(\zeta)$ is almost zero for $-1/4 \leq \zeta \leq 0$ but then it rapidly increases and tends to 1/2 as $\zeta \to -\infty$.
- The gap $\delta(\zeta)$ is almost zero for $\zeta \leq -1/2$ and then slowly increases to ≈ 0.193 for RG.
- Conclusion: $\zeta_c = -1/2$ for our model on RG.



- Order parameter $\mathcal{C}(r, \zeta)$ with link density r for ζ = 1/2, 1/4, 0, -0.1, -1/4, -1/2, -1, -2, -3, -4 and -5, ζ values decreasing from left to right, for RG of $N = 4096$.
- The asymptotic values of the gap $\delta(\zeta)$, the percolation threshold $r_c(\zeta) - 1/2$ and the largest jump $g(\zeta)$ of the Order Parameter plotted with ζ.

Cluster Size Distribution

- Our model: Scaled cluster size distribution $P(s,p)$ at $\zeta = -1$.
- $P(s,p)\Delta p^{-x_1} \sim \mathcal{G}(s\Delta p^{x_2})$ with $x_1 = x_2 = 1.77(5)$.
- Gamma distribution $\mathcal{G}(x) \sim x^a \exp(-bx)$ with $a = 0.93(10)$ and $b = 4.09(10)$.
- EP: $P(s,p)$ follows a Power law as in Ordinary Percolation and the exponent approaches to -2.04(1) for SL and $\approx -2.11(2)$ for RG.



- Cluster size distribution $P(s,p)$ at $\zeta = -1$ (with $p_c = 0.549(2)$) for L= 64 and 128; p = 0.48, 0.50 and 0.52. The solid line is a fit to the Gamma distribution.
- The binned average cluster size distribution $P(s,p)$ of EP; p = 0.49, 0.51, 0.52 and 0.525 for L =1024 (from left to right).

Two other quantities

- The average ratio $\mathcal{R}(p) = \langle s_m / s_{nm} \rangle$ where s_{nm} is the size of the second largest cluster.
- $\mathcal{R}(p)$ for SL with $L = 128$ and AP on SL. For all $\zeta < 0$ the plots are indeed horizontal lines up to $p = p_c(\zeta)$.
- It implies that s_m and s_{nm} have comparable values
- In comparison $\mathcal{R}(p)$ for $\zeta = 0$ and AP on SL grows continuously
- Decay of the fraction $n_1(p, \zeta)$ of isolated nodes in the system.
- In our model, the $n_1(p, \zeta)$ decays very fast and vanishes at or before p_c for all $\zeta < 0$
- $n_1(p)$ helps to predict if the percolation transition is likely to be first order or continuous.

Our model: $\zeta \neq 0$ CPU $\sim N^3$ (RG) and $\sim N^2$ (SL); but CPU $\sim N$ in AP. This limited us to $L = 512$ (SL) and $N = 4096$ (RG).

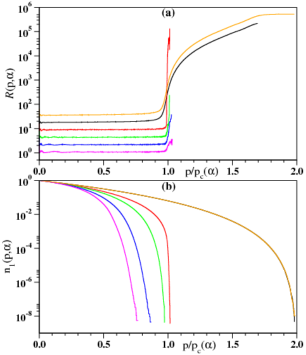
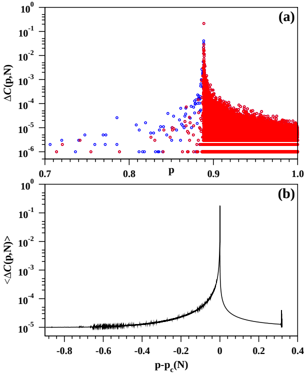


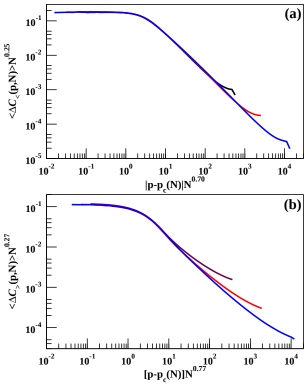
Figure: (a) Ratio $\mathcal{R}(p, \zeta)$ of the sizes of largest cluster and the next largest with $p/p_c(\zeta)$ for $\zeta = -3, -2, -1, -1/2, 0$ and AP on SL from bottom to top. For more visibility y axes have been multiplied by 1, 2, 4, 8, 16 and 32 respectively. (b) Fraction $n_1(p, \zeta)$ of isolated nodes with $p/p_c(\zeta)$ for the same ζ values (from right to left) as in (b).

Sequence of jumps in the Order Parameter for AP



(a) (Blue) For a single run, the sequence of jumps sizes in the Order Parameter have been plotted with the percolation probability p. (Red) The subset of jumps where the largest cluster remains the same.
(b) The jump size averaged over many runs.

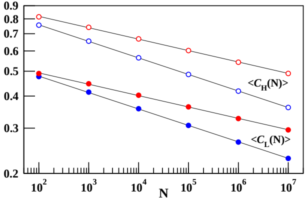
• FINITE-SIZE SCALING OF AV. JUMP SIZE OF OP



For AP the finite-size scaling of average size of the jump in OP for $N = 10^4$ (blue), 10^5 (red) and 10^6 (green). (a) Sub-critical region. The average jump size $\langle \Delta \mathcal{C}_<(p, N)\rangle$ is plotted against deviation $|p - p_c(N)|$. The average slopes give $\alpha_<$ = 1.18(5). (b) Super-critical region. The average jump size $\langle \Delta \mathcal{C}_>(p, N)\rangle$ is plotted against deviation $p - p_c(N)$. The average slopes give $\alpha_>$ = 1.02(5).

• AVERAGE VALUE OF OP AT THE MAXIMAL JUMP

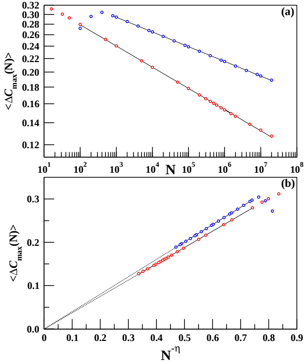
• Maximal jump in the largest component occur when largest component merges with the maximal of the second largest component.
• We assume that the sizes of the largest and the 2nd largest components grow with same power χ at the maximal jump of the OP. That is $\langle s_{max} \rangle \propto N^{\chi}$ for both components.
• Therefore both $\langle \mathcal{C}_L \rangle$ and $\langle \mathcal{C}_H \rangle$ are $\propto N^{1-\chi}$.



The average values of $\langle \mathcal{C}_L(N) \rangle$ (filled circles) and $\langle \mathcal{C}_H(N) \rangle$ (empty circles) are plotted against N for AP (blue) and for da Costa model (red). From slopes the growth exponent χ of the largest component are estimated as 0.9355(5) for AP and 0.9553(5) for da Costa model.

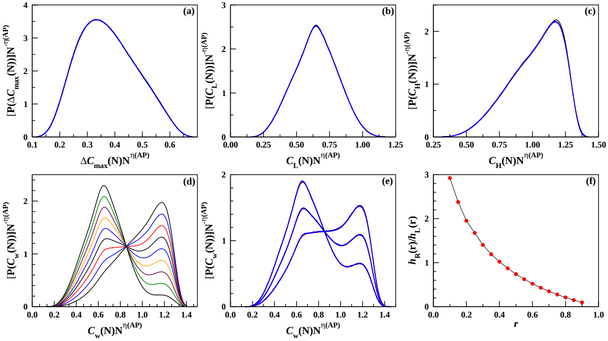
- AVERAGE MAXIMAL JUMP OF OP

- $\langle \Delta \mathcal{C}_{max} \rangle (N)$ = (av. size of the maximal of the 2nd largest comp.)/$N \sim N^{\chi - 1} = N^{-\eta}$ where $\eta = 1 - \chi$.



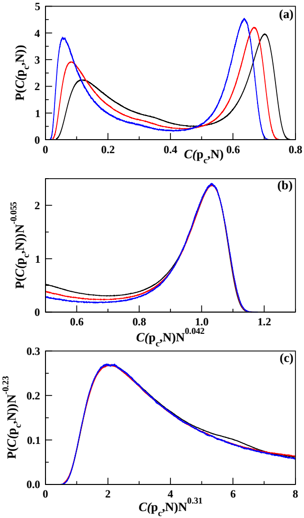
(a) The average maximal jump $\langle \Delta \mathcal{C}_{max}(N) \rangle$ in OP plotted on a log $-$ log scale against graph size N. The slopes are η(AP)≈ 0.0647 (red) and η(da Costa)≈ 0.0449 (blue). (b) The same sets of data are plotted against $N^{-\eta}$ on a lin-lin scale. Continuous lines are straight line fits of the data which are then extrapolated to $N \to \infty$ to meet $\langle \Delta \mathcal{C}_{max}(N) \rangle$ axis at $\langle \Delta \mathcal{C}_{max}(\infty) \rangle \approx -0.0005$ for AP (red) and ≈ -0.0003 for da Costa model (blue).

- FINITE SIZE SCALING WITH SINGLE EXPONENT



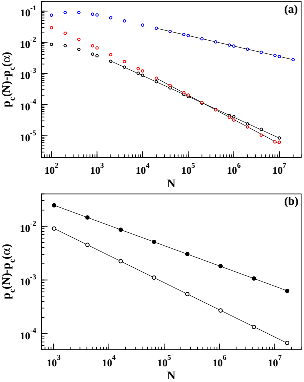
FSS of different probability distributions of the Order Parameter $\mathcal{C}$ of AP using η(AP)$=0.0645$ and with $N=10^3$ (black), 10^4 (red) and 10^5 (blue). (a) Maximal jump sizes $\Delta\mathcal{C}_{max}(N)$. (b) The lower end value $\mathcal{C}_L(N)$ of the maximal jump in $\mathcal{C}$. (c) The higher end value $\mathcal{C}_H(N)$ of the maximal jump in $\mathcal{C}$. (d) The weighted average $\mathcal{C}_W(N)$ of $\mathcal{C}_L(N)$ and $\mathcal{C}_H(N)$ with probabilities r and $1-r$. The right-top curve is for r=0.1 and increased to 0.9 at the interval of 0.1 using only one system size $N=10^3$. (e) The scaling of $P(\mathcal{C}_W(N))$ for r = 0.3 (right-top), 0.5 and 0.7. (f) The ratio of heights of the right peak $h_R(r)$ and left peak $h_L(r)$ of the bimodal distributions in (d) plotted against probability r.

- DIST. OF OP AT $p_c(N)$



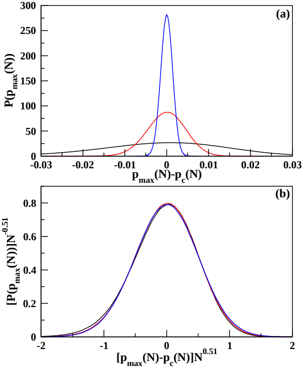
Distribution of the Order Parameter $\mathcal{C}(p_c, N)$ right at the percolation threshold $p_c(N)$. (a) $P(\mathcal{C}(p_c, N))$ plotted against $\mathcal{C}(p_c, N)$ for $N = 10000$ (black), 32000 (red) and 100000 (blue). Finite-size scaling of the right peak in (b) and that of the left peak in (c).

• HOW $p_c(N)$ APPROACHES $p_c(\infty)$



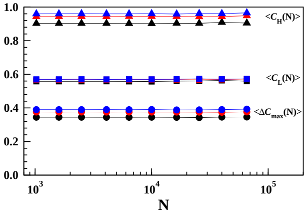
Plot of $p_c(N) - p_c(\infty)$ vs. N on the $\log - \log$ scale for AP (black), da Costa model (red) and Random Graphs (blue). (a) Apart from few small size graphs the data points fit nicely to straight lines giving values of the ν exponent in Eqn. (6) as 1.50, 1.26 and 2.98 respectively. (b) Similar data for the AP (empty circles) and ordinary bond percolation (filled circles) on square lattice giving ν = 1.98 and 2.67 respectively.

• PROB. DIST. OF THE POSITION OF THE MAXIMAL JUMP



For AP the probability distributions $P(p_{max}(N))$ vs. $p_{max}(N) - p_c(N)$ for three system sizes $N = 10^3$ (black), 10^4 (red) and 10^5 (blue) in (a) and their finite-size scaling in (b).

• STRICTLY GLOBAL RULE: AVERAGE OF OP AT THE TWO ENDS OF THE MAXIMAL JUMP



For the Strictly global rule EP model on square lattice the values of $\langle \mathcal{C}_L(N) \rangle$ (square), $\langle \mathcal{C}_H(N) \rangle$ (triangle) and $\langle \Delta\mathcal{C}_{max}(N) \rangle$ (circle) are plotted for ζ = -1, -2 and -3.

- COMPARISON OF DIFFERENT EXPONENTS

Table 1

	AP	da Costa	AP in 2*d*	BP	RG
p_c	0.888449(2) [1]	0.923207508 [3]	0.526565 [2]	1/2	1/2
η_+	0.0402(15) [1]	0.0255(80) [1]	0.018(2) [1]		
η	0.0645(5)	0.0446(5)	0.0217(5)	5/96	1/3
η_-	0.270(7) [1]	0.300(5) [1]	0.078(7) [1]		
χ	0.9355(5)	0.9554	0.9783(5)	91/96	2/3
ν	1.50	1.26	1.98	8/3	2.98

Table: Values of different exponents available in the literature as well as measured in this work. Some known results of Bond Percolation (BP) in 2*d* and for Random Graphs are also included for comparison. The p_c, η_+ and η_- values are taken form [1].

Table 2

ζ	p_c	η	χ	ν
-1	1.00(1)	0.0	1.0	
1/4	0.346(1)	0.44(2)	0.56(2)	1.77(2)

Table: Different exponents of the Strictly Global model in [4] for two values of the parameter ζ.

References:

[1] P. Grassberger, C. Christensen, G. Bizhani, S-W. Son and M. Paczuski, Phys. Rev. Lett. **106**, 225701 (2011).
[2] R. M. Ziff, Phys. Rev. E **82**, 051105 (2010).
[3] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, Phys. Rev. Lett. **105**, 255701 (2010).

Conclusion

- The value of the growth exponent χ ($\langle s_{max} \rangle \sim N^{\chi}$) determines the nature of transition: $\chi < 1$ or, $\chi = 1$ determines if the transition is continuous or, discontinuous.
- In Euclidean space $\chi = D/d$, D being the fractal dimension of the i.i.c. in d dimensions.
- We conjecture that for all 'Limited non-local' and 'Strictly global' rule models $\chi < 1$ and $\chi = 1$ and therefore these models have continuous and discontinuous transitions respectively.

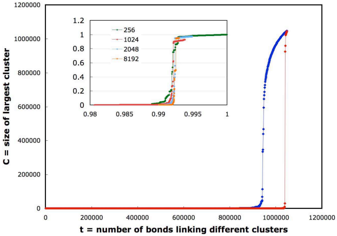
A STRICTLY GLOBAL RULE MODEL

- Single edges between pairs of clusters are randomly occupied with probabilities $\propto (s_i s_j)^{\zeta}$.
- Numerical evidence: for all $\zeta < \zeta_c$ the transition is discontinuous; $\zeta_c = 0$ for SL but -1/2 for RG.
- The effect of the biased occupation rule discouraging growth of large clusters and encouraging growth of small clusters is distinctly visible in the Gamma distribution of the cluster sizes.
- This is in contrast to power law distributions in AP and in other 'Limited non-local' rule models.

• S. S. Manna and A. Chatterjee, Physica A, **390**, 177 (2011).
• S. S. Manna, arXiv:1106.2604.

Loop-less Percolation

- Percolation where links are forbidden to connect two nodes of the same cluster.
- With loop-less condition each cluster is a Tree Graph. Number of clusters: $n = N - t$.
- On square lattice: $p_c = (7 - 3\sqrt{3})/4 \approx 0.450962$ with $p = t/(2N)$.
- $p_c \approx 0.4963$ (Loop-less) and 0.526565 (with Loops).



- Reg. perc. (blue) and PR perc. (red) on $L = 1024$. PR plot shows delayed and explosive growth. Inset is a plot for C/N vs. t/N for L=256, 1024, 2048 and 8192.

[1] R. M. Ziff, Phys. Rev. Lett. **103**, 045701 (2009).

[2] S. S. Manna and B.Subramanian, Phys. Rev. Lett. **18**, 3460 (1996).

Explosive Percolation on Square Lattice [1]

- LL Perc.: Gap $\Delta = t_1 - t_0$. Num. results: $\Delta/N \sim L^{-0.383} = N^{-0.192}$
- t_1 converges rapidly to t_c.
- The main variation of Δ is due to t_0.
- Typical cluster mass $s^* \sim |p - p_c|^{-1/\sigma}$, with $\sigma = 36/91 \approx 0.3956$ in $2d$. Now $s^* = N^{1/2} = L$ corresponding to t_0.
- $|\Delta/N| = |t_0 - t_1|/N \sim |p_0 - p_c| = s^{*-\sigma} = N^{-\sigma/2} = L^{-36/91}$.
- In comparison EP on SL: $\Delta/N \sim L^{-0.683} \sim N^{-0.342}$; $p_c(LLEP) \approx 0.4963$.

[1] R. M. Ziff, Phys. Rev. Lett. **103**, 045701 (2009).