Coloring Simple Hypergraphs

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Problem

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If true, then sharp by letting

$$S = \{(x, y) : x = 1 \text{ or } y = 1\} \setminus (1, 1).$$

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S – collection of n points in general position in the unit square

$$T(S) = \text{area of smallest triangle}$$

$$T(n) = \max_S T(S)$$

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- $T(n) > \frac{1}{2(n-1)^2}$

Conjecture: $T(n) = \Theta(1/n^2)$

Upper Bounds

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Lower Bound

Komlós-Pintz-Szemerédi (1982) $$T(n) > \frac{c\log n}{n^2}$$

Number Theory - Infinite Sidon Sets

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Conjecture (Erdős) $n^{1/2-\epsilon}$

Coding Theory

Fix $k, r \geq 2$. Let A be an $n \times M$ matrix over Z_2 with

- k one's in each column
- every r columns linearly independent over Z_2 (i.e. every set of at most r column vectors does not sum to 0)

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$M := M(n, k, r) =$ maximum number of columns in A

In other words, M is the maximum length of a binary linear code with minimum distance at least $r + 1$ and parity check matrix with n rows and every coordinate having at most k check equations.

Lefmann-Pudlak-Savický (1997)

$$M(n,k,r) > cn^{\frac{kr}{2(r-1)}}$$

Results of Frankl-Füredi on union closed families yield

$$M(n,k,4) < cn^{\frac{\lceil 4k/3 \rceil}{2}},$$

so when $k \equiv 0 \pmod 3$, $\quad M(n,k,4) = \Theta(n^{\frac{2k}{3}})$

Kretzberg-Hofmeister-Lefmann (1999)
If $r \geq 4$ is even, $\gcd(r-1,k) = 1$, then

$$M(n,k,r) > cn^{\frac{kr}{2(r-1)}} (\log n)^{\frac{1}{k-1}}.$$

Naor-Verstraëte (2009) Improvements for different ranges of k, r; connections to extremal graph theory

Independent Sets in Hypergraphs

Let $k \geq 2$ be fixed, $n \to \infty$

Fact (Turán's theorem) Let H be a k-uniform hypergraph with average degree d. Then

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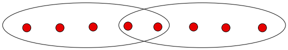
Sharp. Let $H = K_n^k$, then $d = \binom{n-1}{k-1} = \Theta(n^{k-1})$, $d^{\frac{1}{k-1}} = \Theta(n)$ and

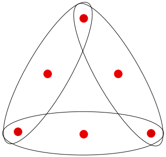
$$\alpha(H) = k - 1 = \Theta(1)$$

What if H is locally sparse?

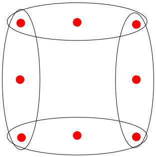
2−cycle



3−cycle



4−cycle



girth g – no cycle of length less than g

simple or linear – girth 3 or no 2-cycle

Theorem (Komlós-Pintz-Szemerédi $k = 3$,
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Let $k \geq 3$ be fixed. Let H be a k-uniform hypergraph with girth at least 5 and (average) maximum degree Δ. Then

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Conjecture (Spencer 1990), Theorem (Duke-Lefmann-Rödl 1995)

Same conclusion holds as long as H is simple.

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- $$T(n) > \frac{c \log n}{n^2}$$
- $|S| > c(n \log n)^{1/3}$ (Improved by Ruzsa)
- $M(n,k,r) > cn^{\frac{kr}{2(r-1)}} (\log n)^{\frac{1}{k-1}}$.

Graph Coloring

$\Delta = \Delta(G) =$ max degree of G

Greedy Algorithm: $\chi(G) \leq \Delta + 1$

Brook's Theorem: $\chi(G) \leq \Delta$ unless $G = K_{\Delta+1}$ or $G = C_{2r+1}$

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Ajtai-Komlós-Szemerédi, Shearer: $\alpha(G) > c\frac{n}{\Delta}\log\Delta$

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New Result

Theorem (Frieze-M) Let $k \geq 3$ be fixed. Then there exists $c = c_k$ such that every k-uniform simple H with maximum degree Δ has

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- The result is sharp apart from the constant c

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- Johansson (1997) additional new ideas for triangle-free graphs
- Vu (2000+) extended Johansson's ideas to more general situations

More Tools

Concentration Inequalities

- Hoeffding/Chernoff
- Talagrand
- Local Lemma
- Kim-Vu polynomial concentration takes care of dependencies

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Example of Kim-Vu: Let $G = G(n, p)$, $p = \frac{1}{\sqrt{n}}$.
Fix a vertex x in G

$T(x)$ is the number of triangles containing x

Then $\mu = E(T(x)) = \binom{n-1}{2} p^3$ but triangle are not independent.
Still

$$P(|T(x) - \mu| > \delta\mu) < e^{-c_\delta \mu}$$

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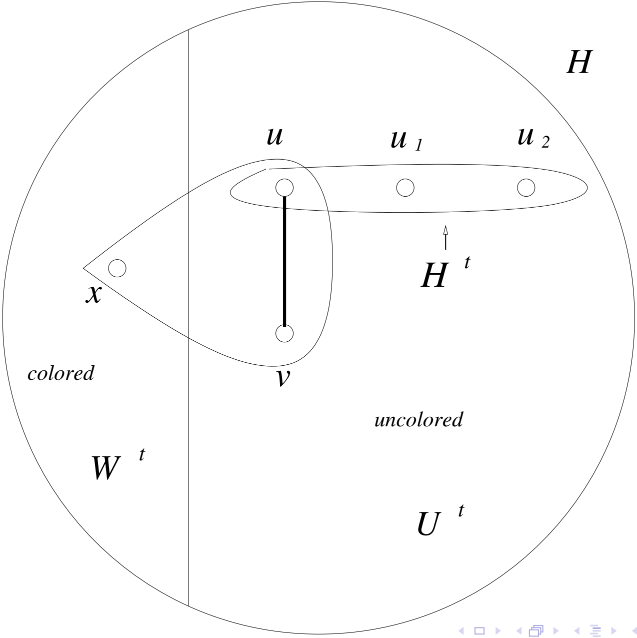
- $p_u^0 = (1/q, \ldots, 1/q)$ – initial color vector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H t
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Parameters p_u are updated in a (complicated) way to maintain certain properties of $H^t = H[U]$

Parameters ($k = 3$)

During the process, we must choose update values to maintain the values of certain parameters:

- $\sum_c p_u(c) \sim 1$
- $e_{uvw} = \sum_c p_u(c)p_v(c)p_w(c) \ll \frac{\log \Delta}{\Delta}$
- $\deg(v) \leq \left(1 - \frac{1}{\log \Delta}\right)^t \Delta \sim e^{-t/\log \Delta}\Delta$
- Also, entropy is controlled; key new idea of Johansson; don't need martingales, Hoeffding suffices

Continue till $t = \log \Delta \log \log \Delta$ and then apply Local Lemma.

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- Major Open Conjecture (Erdős et. al.)

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More Optimism

Conjecture (Frieze-M)

Let F be a fixed k-uniform hypergraph. Then there exists $c = c_F$ such that every F-free k-uniform hypergraph H with maximum degree Δ satisfies

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Algorithms??

Convert our proof to a deterministic polynomial time algorithm that yields a coloring with $c(\Delta/\log \Delta)^{1/(k-1)}$ colors

Moser-Tardos results yield a randomized algorithm

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- Gyárfás-M $\Omega(n)$ if Frieze-M Conjecture holds for $k = 3$ and $F = K_9^3$

Thank You