On the structure of dense H-free graphs

Tomasz Łuczak, Stéphan Thomassé

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Budapest's Question

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Sós'69

Andrásfai, Erdős, Sós'74

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A graph G on n vertices is dense if $\delta(G) \geq an$ for some constant $a > 0$.

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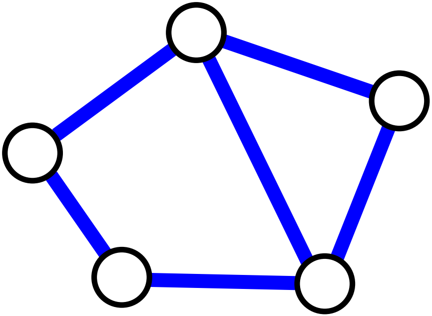
I. We can study the topological structure, i.e. ask if every maximal dense graph is a blow-up of one of just few graphs:



A “topological” approach to the structure

Examples:

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Each maximal triangle-free graph G_n with $\delta(G_n) > 3n/8$ is a blow-up of either K_2 or C_5.

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II. We can study the chromatic properties, i.e. ask whether each dense H-free graph has a small chromatic number.

A “chromatic” approach to the structure

Examples:

For each triangle-free graph G_n with $\delta(G_n) > n/3$ we have $\chi(G_n) \leq 4$.

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Andrásfai, Erdős, Sós’74.
For each K_k-free graph G_n with $\delta(G_n) > \frac{3k-7}{3k-4}n$ we have $\chi(G_n) \leq k-1$.

Cluster points

Each maximal triangle-free graph G_n with $\delta(G_n) > 2n/5$ is bipartite.

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$$\frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \frac{5}{14} \dots \qquad \frac{1}{3}$$

$\nu_\tau(H)$

Definition

$\nu_\tau(H)$ is the smallest $a \geq 0$ for which the following holds:

> *for every $\epsilon > 0$ there exists $f(\epsilon)$ such that each maximal H-free graph G on n vertices with $\delta(G) \geq (a+\epsilon)n$ is a blow-up of a graph with at most $f(\epsilon)$ vertices.*

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$\nu_\tau(H) > 0$ provided $\chi(H) \geq 3$.

A second thought on H-free graphs

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A second thought on H-free graphs

Conjecture

Let G be a C_{2k+1}-free graph with

$$\delta(G) > \frac{2n}{2k+3}\,.$$

Then G is bipartite.

Theorem Győri, Nikiforov, Schelp'03

The above is true for $k = 1, 2, 3, 4$, and false otherwise.

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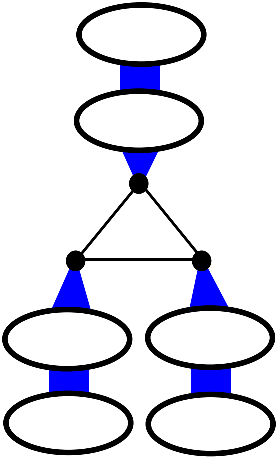

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Theorem Łuczak, Thomassé

For every H either

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Vapnik-Červonenkis dimension

Definition

Let $\mathcal{F}$ be a family of subsets of V. We say that a set $X \subseteq V$ is shattered by $\mathcal{F}$ if for each $Y \subseteq X$ there is an $F \in \mathcal{F}$ such that $Y = F \cap X$.

Vapnik-Červonenkis dimension

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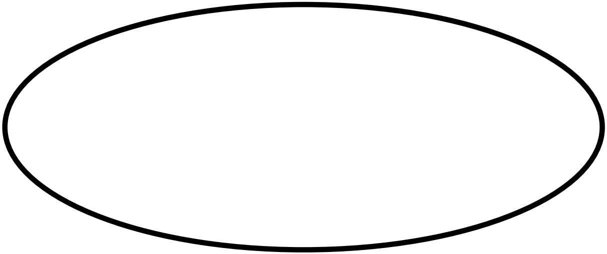

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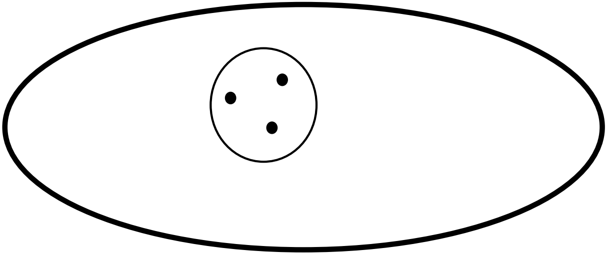

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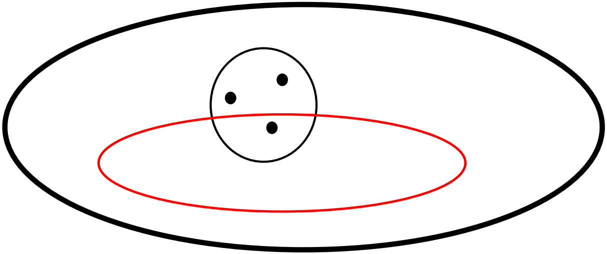

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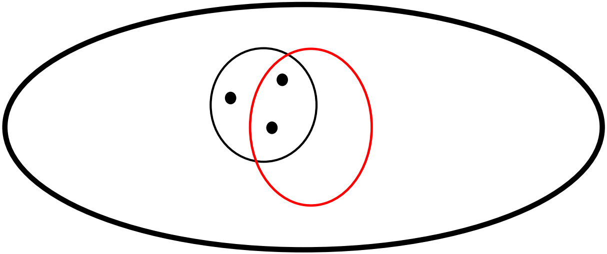

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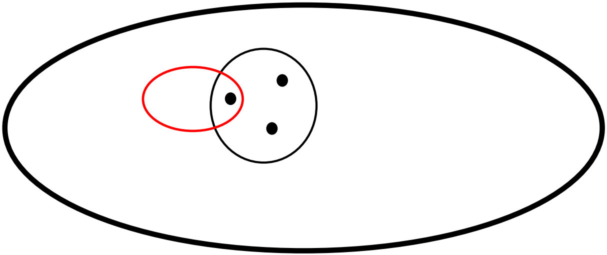

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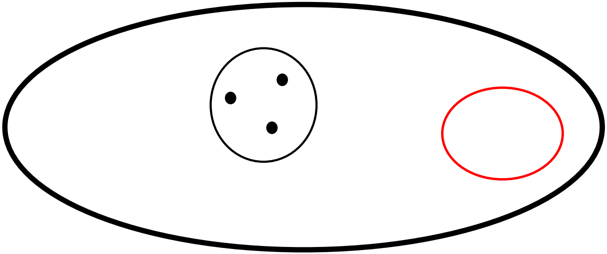

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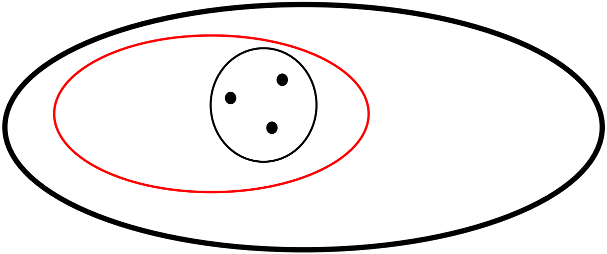

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The VC-dimension of a family of sets $\mathcal{F}$, denoted by $d_{VC}(\mathcal{F})$, is the maximum size of a set shattered by $\mathcal{F}$.

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The VC-dimension $d(G)$ of a graph $G = (V, E)$ is the VC-dimension of the family of sets

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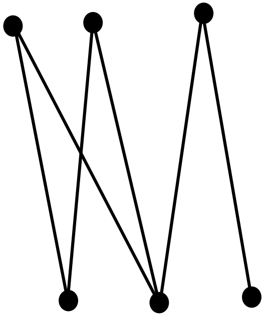
Theorem Haussler, Welzl'87

If G is a graph with vertex set $[n] = \{1, 2, \ldots, n\}$, minimum degree at least an, $a > 0$, and VC-dimension d, then the covering number $\tau(G)$ of G is bounded from above by

$$\frac{32d}{a} \ln\left(\frac{2d}{a}\right).$$

Weakly induced bipartite graphs



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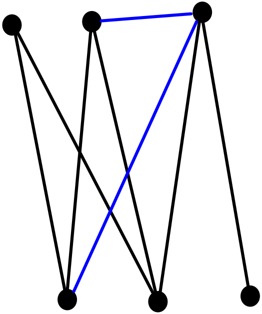

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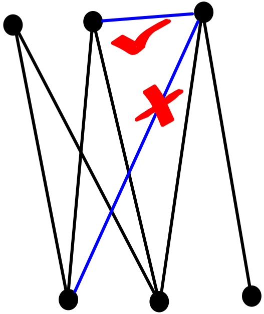

VC-dimension and ν_χ

Theorem Łuczak, Thomassé

Let us suppose that a triangle-free graph G with n vertices and $\delta(G) \geq an$, where $a > 0$, contains no weakly induced copy of some bipartite graph H.
Then, $\chi(G) \leq f(H, a)$.

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Proof Let G be a graph with $\delta(G) \geq an$ without weakly induced copy of H. Find in G a bipartite subgraph G' such that $\delta(G') \geq an/2$.

Proof (cont.)

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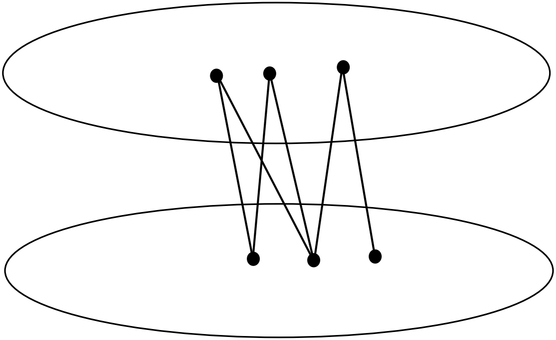

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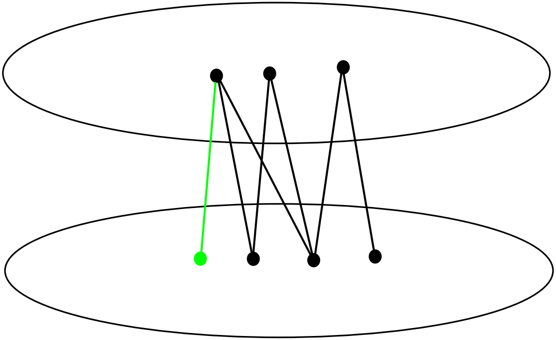

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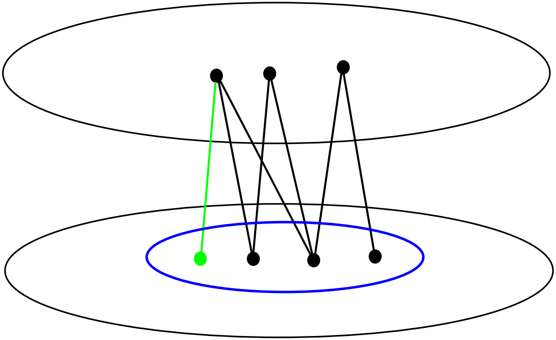

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Hence, $d(G')$ is bounded, and by Vapnik-Červonenkis theorem,

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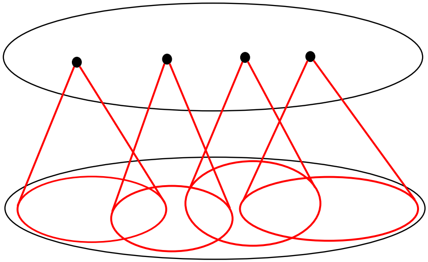

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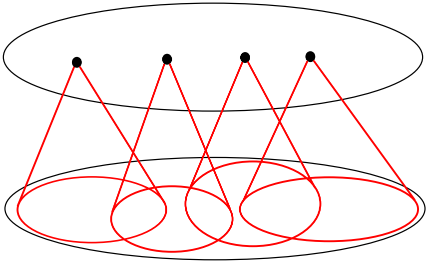

However, since G is triangle-free,

$$\chi(G) \leq 2\tau(G) \leq 2f(H,a)\,. \quad \square$$

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Then, $\chi(G) \leq f(H, a)$.

Lemma Brandt'02

If G is triangle-free and $\delta(G) \geq an$ for some $a > 1/3$, then G contains no weakly induced copy of 4-cube Q_4.

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$\nu_\chi(K_3) \leq 1/3$.

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There exist triangle-free graphs with density close to 1/3 and arbitrary large chromatic number.

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Kneser graph $KG(2m+k,m)$

Definition

$KG(2m+k,m)$ is a graph whose vertices are m-elements subsets of $\{1,2,\dots,2m+k\}$ and two of them are joined by an edge if they are disjoint.

Theorem Lovász'78

$$\chi(KG(2m+k,m)) = k+2\,.$$

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Hajnal-Kneser graph $F(2m+k, m, n)$

Let $n \gg m \gg k$.
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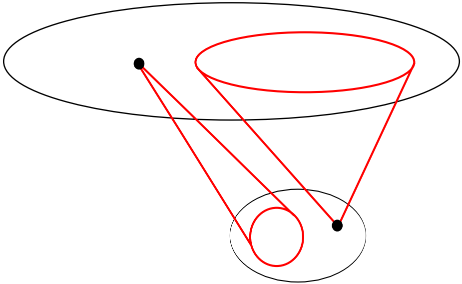

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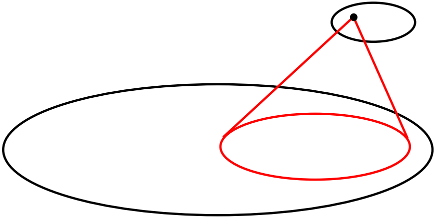

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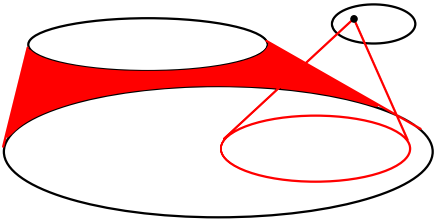
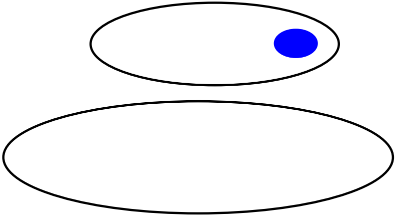
Blow up the lower set to the size roughly $2n/3$.
Finally add an upper independent set of size roughly $n/3$ and connect its vertices with all the vertices of the lower set.



Hajnal-Kneser graph $F(2m+k, m, n)$

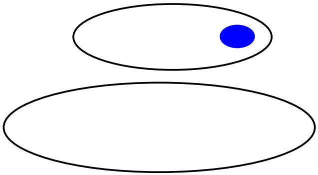
If $n \gg m \gg k$, then $F(2m+k, m, n)$ has n vertices, the minimum degree close to $n/3$, and an unbounded chromatic number which is at least $k+2$ (coming from the small blue subgraph isomorphic to Kneser graph).



Hajnal-Borsuk graph $B(k,\ell,n)$

An analogous geometric construction (roughly speaking one should use a kind of Borsuk graph instead of Kneser graph) gives a similar looking graph $B(k,\ell,n)$, where $\ell \ll k \ll n$ which again has n vertices, the minimum degree close to $n/3$, and an unbounded chromatic number which is at least $k+2$ (coming from the upper part).

But $B(k,\ell,n)$ has also the property that the upper part contain no cycles shorter than ℓ and each odd cycle shorter than ℓ has at least two vertices in the lower part.



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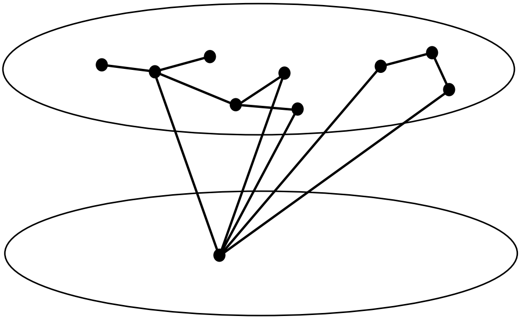
It means that the subgraph induced by the upper half looks locally as a tree, which is a bipartite graph, and each vertex of the lower part can be only adjacent to vertices from one part of the bipartition.



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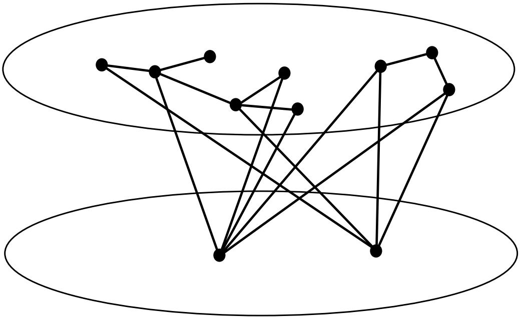

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Theorem **Łuczak, Thomassé**

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If H cannot be homomorphically embedded in $B(k, \ell, n)$ for some ℓ, then, clearly, $\tilde{\nu}_\chi(H) \geq 1/3$.

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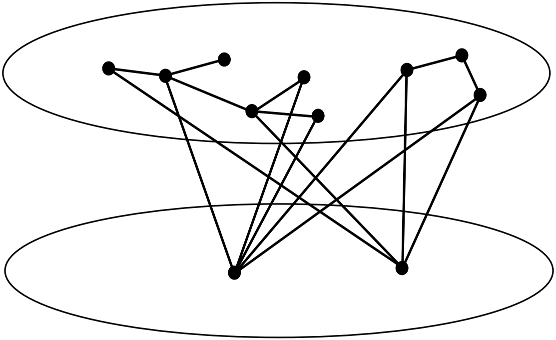
Theorem Łuczak, Thomassé

For every H either $\tilde{\nu}_\chi(H) = 0$, or $\tilde{\nu}_\chi(H) \geq 1/3$.

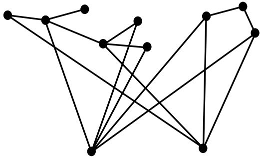
If H cannot be homomorphically embedded in $B(k, \ell, n)$ for some ℓ, then, clearly, $\tilde{\nu}_\chi(H) \geq 1/3$.

Hence, it is enough to show that if H is such that it can be embedded into every $B(k, \ell, n)$, then each H-hom-free graph G with $\delta(G) \geq an$ has a bounded chromatic number.

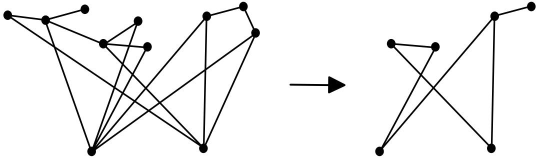
Graphs with $\tilde{\nu}_\chi(H) = 0$



Graphs with $\tilde{\nu}_\chi(H) = 0$

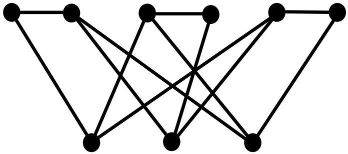


Graphs with $\tilde{\nu}_\chi(H) = 0$



Graphs with $\tilde{\nu}_\chi(H) = 0$

Thus, it is enough to prove that $\tilde{\nu}_\chi(H) = 0$ for graphs H of the following type:

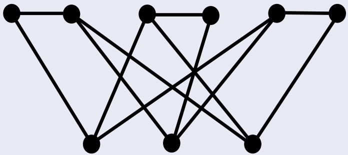


Graphs with $\tilde{\nu}_\chi(H) = 0$

More precisely, we need to show the following statement.

Theorem

If a graph G with $\delta(G) \geq an$, $a > 0$, contains no copies of



then its chromatic number is bounded by $f(a)$.

Generalized Vapnik-Červonenkis dimension

Definition

Let $\mathcal{F}^{(2)}$ be a family of pairs of subsets of V. We say that a set of pairs $\{A_i, B_i\}_{i\in I}$ from $\mathcal{F}^{(2)}$ is complete if for every $J \subseteq I$

$$\bigcap_{j\in J} A_j \cap \bigcap_{\ell\in I\setminus J} B_\ell \neq \emptyset\,.$$

The $VC^{(2)}$-dimension of a family $\mathcal{F}^{(2)}$, denoted by $d^{(2)}_{VC}(\mathcal{F}^{(2)})$, is the maximum size of a complete set of pairs from $\mathcal{F}^{(2)}$.

Generalized Vapnik-Červonenkis dimension

Theorem Łuczak, Thomassé

Let $\mathcal{F}^{(2)}$ be a family of pairs of subsets of $[n]$ such that each subset has size at least an, $a > 0$, and G be a graph whose edges are pairs from $\mathcal{F}^{(2)}$. Then

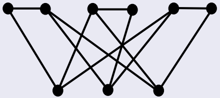
$$\chi(G) \leq f\left(a, d^{(2)}_{VC}\left(\mathcal{F}^{(2)}\right)\right)$$

for some (explicit) function f.

Proof of Main Theorem

Theorem

If a graph G with $\delta(G) \geq an$, $a > 0$, contains no copies of



then its chromatic number is bounded by $f(a)$.

Proof of Main Theorem

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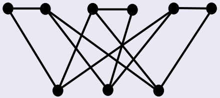
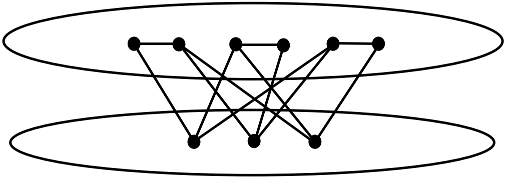


then its chromatic number is bounded by $f(a)$.

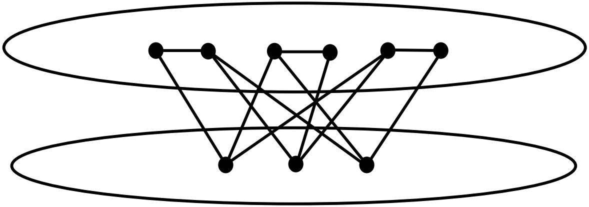
Proof Find a bipartition of G so that each vertex has at least $an/2$ neighbours in the opposite set of the bipartition.

Proof of Main Theorem

Proof Find a bipartition of G so that each vertex has at least $an/2$ neighbours in the opposite set of the bipartition.



Proof of Main Theorem



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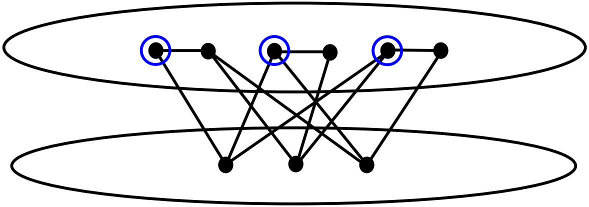

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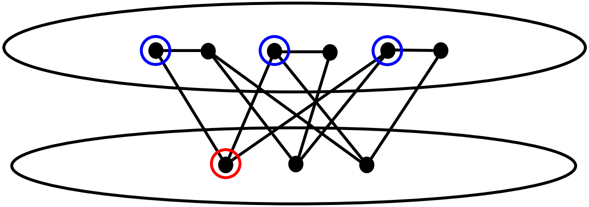

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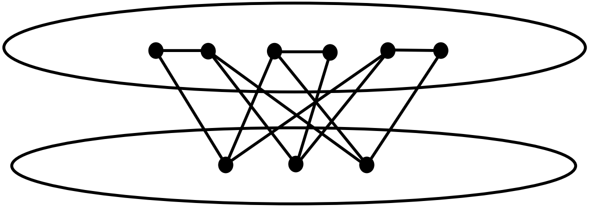

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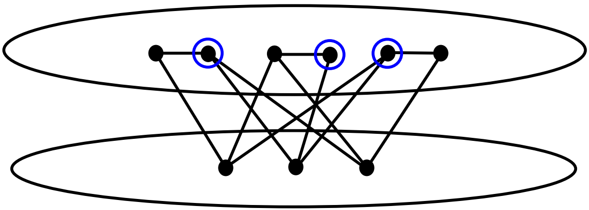

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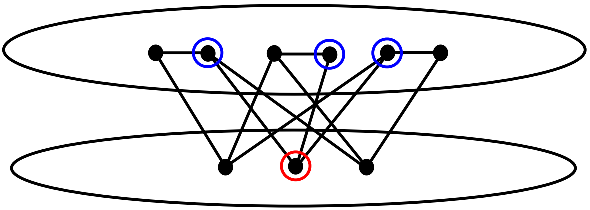

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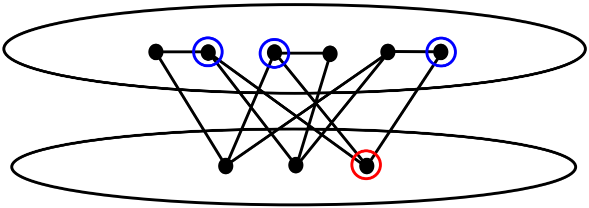


Proof of Main Theorem

Consequently, by our result on $d_{VC}^{(2)}$, the subgraph induced by a lower part has a bounded chromatic number.

Clearly, the same is true for the upper subgraph as well and so the assertion follows. $\square$

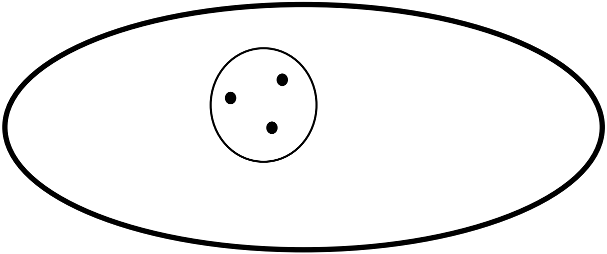
Alternative definition of VC^2-dimension

Definition

Let $\mathcal{F}^2$ be a family of pairs of disjoint subsets of V.
We say that a set $X \subseteq V$ is 2-shattered by $\mathcal{F}^2$ if for each partitions $X = Y \cup Z$ there is an $\{F_1, F_2\} \in \mathcal{F}^2$ such that $Y = F_1 \cap X$ and $Z = F_2 \cap X$.

Alternative definition of VC^2-dimension

Example:



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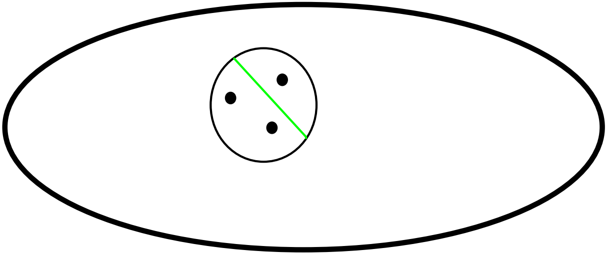

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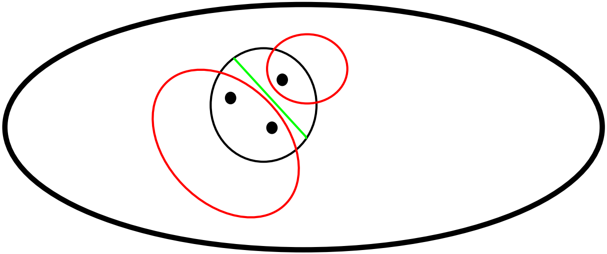

Alternative definition of VC^2-dimension

Definition

The $VC^{(2)}$-dimension of the family of disjoint pairs of sets $\mathcal{F}^2$, denoted by $d_{VC}^{(2)}(\mathcal{F}^2)$, is the maximum size of a set 2-shattered by $\mathcal{F}^2$.

Alternative definition of VC^2-dimension

But we may have $A_i = B_i = [n]$ for all pairs $\{A_i, B_i\}$ from $\mathcal{F}^2$!

Open problems

Problem 1

Does there exist $\eta > 0$ such that for every H we have either $\nu_\chi(H) = 0$ or $\nu_\chi(H) \geq \eta$.

Problem 2

Compute $\nu_\tau(C_{2k+1})$.

Open problems

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Problem 2

Compute $\nu_\tau(C_{2k+1})$.

$1/5 \leq \nu_\tau(C_5) \leq 1/3$.

Open problems

Definition

$\nu_\chi(k)$ is the smallest $a \geq 0$ for which the following holds:

> *for every $\epsilon > 0$ there exists $f(\epsilon)$ such that every graph G on n vertices with $\delta(G) \geq (a+\epsilon)n$ such that the neighbourhood of each vertex of G is k-chromatic is at most $f(\epsilon)$-chromatic.*

Conjecture

$\nu_\chi(2) = 1/2.$

Open problems

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$$\nu_\chi(1) = \nu_\chi(K_3) = 1/3.$$

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Thank you