Multipartite entanglement quantification in weighted graph states

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Abstract

We consider the question of evaluating multipartite entanglement measures for weighted graph states. Weighted graph states are a natural generalisation of the usual graph states used in quantum information processing. For graph states the interaction between subsystems is uniform whereas in the case of weighted graph states it can vary. First we present two different methods of evaluating three multipartite entanglement measures in graph states, namely the Schmidt measure, the relative entropy of entanglement and the geometric measure. One method relies on stabiliser formalism while the other is a new method which we call projected separable pairs states. We then focus on the second method and study whether it can be generalised to the case of weighted graph states.

1 Motivation

Entanglement plays one of the central roles in quantum information and quantum computation. A great deal of effort has been devoted to classifying states according to their entanglement properties [1]. This task has proven to be a difficult one due to the rich structure encountered for states of more than two parties [2]. Trying to classify quantum states according to a coarser picture based on their separability properties is equally hard [3].

Quantification of entanglement in multipartite states is one of the fundamental problems in quantum information theory. A multitude of multipartite entanglement measures exists that aims to achieve this goal. Some have operational meanings arising from certain information processing tasks, whereas others are functions satisfying axiomatic definitions.

We concentrate on three measures of multipartite entanglement: Schmidt measure [4], relative entropy of entanglement [5] and geometric measure [6]. As all three measures are defined as minimisations of distances in Hilbert space or over all linear decompositions into product states they are extremely hard to compute analytically. Examples of states for which any of these measures can be computed are sparse and usually contain come form of symmetry or admit an efficient description that facilitates the evaluation.

One particular class of multipartite quantum states whose entanglement properties can be studied analytically to a high degree are the graph states [7]. Graph states play an important role as resource states in measurement-based quantum computation [8] and in some communication protocols such as quantum secret sharing [9]. Graph states arise very naturally when subsystems of a physical system are allowed to interact via an Ising-type interaction. The entanglement properties have been studied in a number of settings. The Schmidt measure has been analysed for all graph states up 12 qubits in [10], while the relative entropy of entanglement and the geometric measure have been analysed in [11]. A unified picture of these measures in graph states has been presented in [12].

Graph states offer an idealised description of real physical systems interacting via an Ising-type interaction. To go beyond this simplification we have to consider weighted graph states [13]. Unlike in the case of pure graph states the pairwise interaction time differs for various pairs of particles in weighted graph states. So far weighted graph states have found numerous uses in describing various disordered systems such as spin gases [13] as well implementations for a number of quantum information processing tasks such as producing random circuits [14]. Another motivation to study weighted graph states arises from experimental considerations of creating graph states. Entangling gates between qubits needed to create graph states cannot be implemented with perfect accuracy and so the resulting state will be more accurately modelled by a weighted graph state.

The entanglement properties of these interesting states has been studied in [13]. But so far these studies were limited to the case of bipartite entanglement. In this work we address the question whether multipartite measures of entanglement can be easily evaluated for these states and highlight some of the difficulties encountered in doing so.

2 Methods and Results

In order to tackle the weighted graph states we first develop methods of evaluating the above-mentioned entanglement measures in pure graph states. We map the problem of evaluating these measures to a single problem in graph theory, namely the problem of finding the maximum independent set. This problem is a well known NP-complete problem, [15], and has been studied intensively [16]. We prove that if the maximum independent set can be found we can automatically construct the minimal linear decomposition into product states of the corresponding graph state as well as its respective closest separable and closest product states for a large class of pure graph states. Therefore we can evaluate the three entanglement measures. This approach also highlights why it is so difficult in general to evaluate multipartite entanglement measures in pure graph states by making a direct connection between the procedure of minimising distances in Hilbert space to an NP-complete problem from graph theory.

We present two methods of evaluating the entanglement measures. The first method utilises properties of graph state stabilisers to find the general N-qubit Schmidt decomposition of a given graph state $|G\rangle$. Doing this requires knowledge of the maximum independent set. Using this decomposition we can immediately evaluate the Schmidt measure and construct the closest separable and closest product states, effectively permitting us to calculate the relative entropy of entanglement and the geometric measure.

The second method is inspired by a particular description of quantum systems called the projected entangled pairs states [17]. In this description a physical qubit is replaced by a number of virtual qubits, the exact number depending on the degree of the qubit.

Each virtual qubit interacts with one different virtual qubit at a different physical site. Together they are represented by a Bell pair. Finally to obtain the desired quantum state the virtual qubits are projected into lower-dimensional subspaces at each physical site.

We adapt this method to describe closest separable states, which we call projected separable pairs states. To do this we use two particular forms of 2-qubit virtual closest separable state as our basic building blocks, placing them at each edge of the graph state. The exact placement can be obtained from graph theory. As before we project the virtual qubits at physical sites into lower dimensional subspaces to obtain the desired separable state.

A natural question to ask at this point is whether the second method is necessary since we already have a procedure of obtaining the closest separable state using stabiliser formalism. It turns out that if one considers weighted graph states the previous stabiliser method does not have a straightforward generalisation. The main obstacle lies in the fact that there is no known method of obtaining the general Schmidt decomposition for weighted graph states of more than 2 qubits. This is precisely why we have developed the projected separable states pairs description of separable states.

We were able to construct 2-qubit closest separable states which can be used in a similar fashion as previously to construct a separable state of virtual qubits that closely mimics the geometry of the weighted graph state. Applying particular projectors at physical sites then produces the desired separable states that can be used to quantify relative entropy of entanglement in pure weighted graph states.

The difficulty now lies in finding the correct form of the projectors that produces a separable state that minimises the relative entropy of entanglement. This is more complicated for weighted graph states than for their non-weighted counterparts. In the case of graph states we obtained the closest separable state which allowed us to tailor the projector to produce these states. In the case of weighted graph states we do not have such knowledge. Therefore any "closest" separable state obtained in such a way will tell us about an upper bound on relative entropy of entanglement unless one can show that it saturates some lower bound. Despite this shortcoming of this method it is a first step towards quantifying multipartite entanglement in pure weighted graph states.

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