Entanglement in the Grover's Search Algorithm

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I. INTRODUCTION AND BACKGROUND

Entanglement is a phenomena by virtue of which two quantum particles in spite of being separated from each other by long distances, retain some correlations. Entanglement is perceived as a resource which facilitates faster and more secured communication as compared to classical means [1]. It is believed to be the primary reason behind the speed up achieved by quantum algorithms over their classical counterparts [2]. However, the lack of a mathematical structure for higher qubits make the study of entanglement difficult [3].
The presence of entangled states across various iterations of the quantum search algorithm (quantum algorithm that searches one or more entries from a large unstructured database quadratically faster than the best known classical algorithm) has been detected [6–10]. However, the nature of entanglement in the algorithm is yet unclear and this is precisely what we analyse. Our work reveals how entanglement varies in the quantum search algorithm across iterations for any arbitrary size of the database (denoted by N) and for any number of marked states (denoted by M). The variation of entanglement with the changes in N and M shows that the maximum value of entanglement is monotonous with the number of qubits n (where $N = 2^n$).
In general, for the Grover's algorithm after the r^{th} iteration the state,

$$|\psi_r\rangle = G^r|\psi_0\rangle = \frac{\cos\theta_r}{\sqrt{2^n - M}}|S_0\rangle + \frac{\sin\theta_r}{\sqrt{M}}|S_1\rangle. \quad (1)$$

Here, $|S_0\rangle$ is the superposition of the non-marked states, while $|S_1\rangle$ is the superposition of all the target states. At the r^{th} iteration, $\theta_r = (r + \frac{1}{2})\sin^{-1}(2\sqrt{\frac{M}{N}})$ [4, 5] and the algorithm iterates $r_{opt} = O(\sqrt{N/M})$ times.

II. OBTAINING ENTANGLEMENT IN THE GROVER'S ALGORITHM

In order to obtain the nature of entanglement in the quantum search algorithm, the entanglement of state $|\psi_r\rangle$ needed to be calculated. Clearly, we were dealing with multipartite entanglement and the requirement was to make use of a multipartite entanglement measurement criteria. For this, we made use of the geometric measure of entanglement.
The geometric measure of entanglement of a state $|\psi\rangle$ is expressed as its distance from its nearest separable state $|\zeta\rangle$. In other words, the overlap between $|\psi\rangle$ and $|\zeta\rangle$ is maximized and the entanglement of the state $|\psi\rangle$ is expressed as [11]

$$E(|\psi\rangle) = 1 -_{max_\zeta} |\langle\zeta|\psi\rangle|^2. \quad (2)$$

Thus, to calculate entanglement of $|\psi_r\rangle$, we needed to obtain the inner product and $\langle\zeta|\psi_r\rangle$ and maximize the same. The nearest fully separable state is assumed to be

$$|\zeta\rangle = (\cos\frac{\phi}{2} + e^{i\gamma}\sin\frac{\phi}{2})^{\otimes n} \quad (3)$$

While maximizing the overlap, $\gamma = 0$, as the amplitudes of $|\psi_r\rangle$ are positive. An important observation was that $|\zeta\rangle$ is permutation invariant and the coefficients of all basis states in $|\zeta\rangle$ containing the same number of 0's and 1's are equal. Thus for all basis states with $n - k$ zeroes and k ones, the coefficient is $\cos^{n-k}\frac{\phi}{2}\sin^k\frac{\phi}{2}$ and there are $\binom{n}{k}$ basis states having this coefficient.
Using these two observations, the generalized expression for entanglement of $|\psi_r\rangle$ can be calculated.

III. RESULTS

A. Analytical Results

The generalized expression for entanglement of $|\psi_r\rangle$ for n qubits and M marked states such that the first marked state has n_1 zeroes, the second n_2 zeroes...., the M^{th} contains n_M zeroes, comes out to be

$$\begin{aligned} E(|\psi_r\rangle) = 1 -_{max_\phi} |&\frac{\cos\theta_r}{\sqrt{N-M}}(\cos\frac{\phi}{2} + \sin\frac{\phi}{2})^n + \\ &(\frac{\sin\theta_r}{\sqrt{M}} - \frac{\cos\theta_r}{\sqrt{N-M}})(\sum_{i=1}^{M}\cos^{n-n_i}\frac{\phi}{2}\sin^{n_i}\frac{\phi}{2})|^2. \end{aligned} \quad (4)$$

This gives the entanglement at the r^{th} iteration of the algorithm. Assuming ϕ_r to be the value of ϕ corresponding to the maximum overlap, we obtain a bound for the entanglement at the r^{th} iteration

$$E(|\psi_r\rangle) \leq 1 - \frac{(s_1 - s_2)^2}{N-M} - \frac{s_2^2}{M}. \quad (5)$$

Here, $s_1 = (\cos\frac{\phi_r}{2} + \sin\frac{\phi_r}{2})^n$ and $s_2 = \sum_{i=1}^{M}\cos^{n-n_i}\frac{\phi_r}{2}\sin^{n_i}\frac{\phi_r}{2}$.

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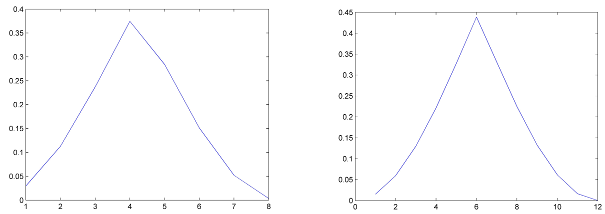
B. Numerical Results

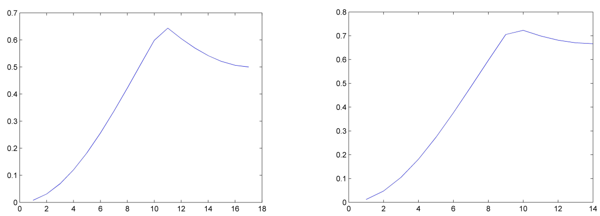
When M and N are changed, the dynamics of entanglement reveal interesting features. In the first scenario, when only one marked state is considered, the plot in Figure 1 reveals that the maximum value of entanglement is always obtained at exactly half of the optimal number of iterations and increases with increase in the number of qubits. When M is fixed and n changes, as shown in



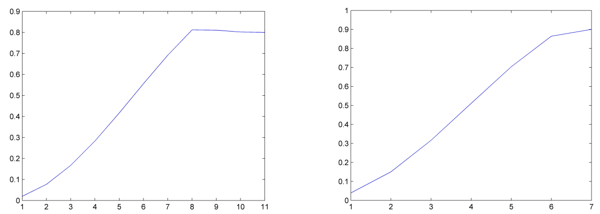
(a)Entanglement for n=7 qubits (b)Entanglement for n=8 qubits

FIG. 1: Entanglement with respect to the number of iterations when $M = 1$ and $|00..0\rangle$ is marked. Here Y-axis depicts the entanglement while the number of iterations is shown along the X-axis.

Fig. 2, we observe that the peak value of entanglement increases with increase in M. Also, interestingly, with an increase in M, the rise in entanglement decreases and it takes longer (more number of iterations)to reach the peak, or in other words, the peak shifts to the right. This has been depicted in Table I.



(a)Entanglement when M=2 (b)Entanglement when M=3



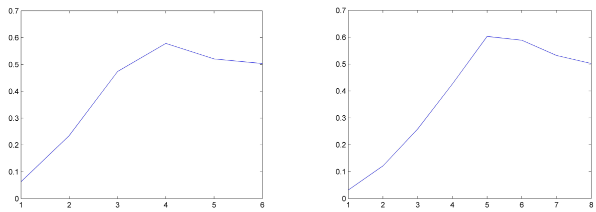
(c)Entanglement when M=5 (d)Entanglement when M=10

FIG. 2: Entanglement with respect to the number of iterations when $n = 10$ and M changes. Here entanglement is plotted along the Y-axis while the number of iterations is plotted along the X-axis.

The dynamics of entanglement in the quantum search algorithm is also revealed when M is fixed and N changes. For this, we have assumed scenarios where the algorithm converges to known physical states namely the n-qubit GHZ state and W-states as shown in Figure 3 and Figure 4 respectively. In this scenario too, the entanglement increases with the number of qubits before the entanglement converges to that of GHZ state or W state.

No. of Marked states	Optimal no. of iterations	No. of iterations required to reach peak entanglement
1	24	$0.5r_{opt}$
2	17	$0.647r_{opt}$
3	14	$0.714r_{opt}$
⋮	⋮	⋮
5	11	$0.727r_{opt}$
⋮	⋮	⋮
10	7	r_{opt}

TABLE I: Iterations required to attain maximum entanglement for $n = 10$ qubits



(a)Entanglement for n=7 qubits (b)Entanglement for n=8 qubits

FIG. 3: Entanglement dynamics with respect to the number of iterations when $|00..0\rangle$ and $|11..1\rangle$ are marked. Here Y-axis depicts entanglement and the number of iterations is shown in the X-axis.

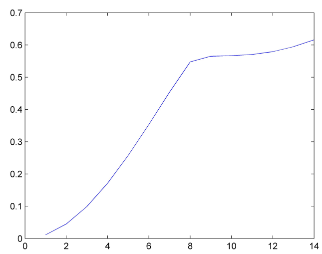


FIG. 4: Entanglement dynamics with respect to the number of iterations when $n = 12$ and the target state is a W state. Here entanglement is plotted along the Y-axis and the number of iterations along the X-axis.

IV. CONCLUSION

Our work analyses entanglement in the quantum search algorithm using geometric measure of entanglement. The dynamics of entanglement for various scenarios have been revealed which contributes to the better understanding of the role of entanglement in quantum algorithms.
For a more detailed version of our work kindly refer to arxiv:1305.4454v2.

[1] D. Bruβ, J. Math. Phys. **43**, 4237 (2002).
[2] A. Galindo and M. A. Martin-Delgado, Rev. Mod. Phys. **74**, 347 (2002).
[3] S. Aaronson and D. Gottesman, Phys. Rev. A **70**, 052328 (2004).
[4] M. A. Nielsen and I. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2002), p 11, 95.
[5] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
[6] D. Deutsch and R. Jozsa, Proc. R. Soc. Lond. A **439**, 553 (1992).
[7] R. Orus and J. I. Latorre, Phys. Rev. A **69**, 052308 (2004).
[8] A. Galindo and M. A. Martin-Delgado, Phys. Rev. A **62**, 62303 (2000).
[9] D. A. Meyer, Phys. Rev. Lett. **85**, 2014 (2000).
[10] D. A. Meyer and N. R. Wallach, J. Math. Phys. **43**, 4273 (2002).
[11] T. C. Wei and P. M. Goldbart, Phys. Rev. A **68**, 042037 (2003).